THE METHOD OF FROBENIUS TO FUCHSIAN PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. To Fuchsian partial differential equations in the sense of M.S. Baouendi and C. Goulaouic, which is a natural extension of ordinary differential equations with regular singularity at a point, all the solutions in a complex domain are constructed along the same line as the method of Frobenius to ordinary differential equations, without any assumptions on the characteristic exponents. The same idea can be applied to Fuchsian hyperbolic equations considered by H. Tahara.

1. Introduction

Let $\boldsymbol{C}$ be the set of complex numbers, t be a variable in $\boldsymbol{C}$, and $x = (x_1, \dots, x_n)$ be variables in $\boldsymbol{C}^n$. We consider a Fuchsian partial differential operator with weight 0 defined by M. S. Baouendi and C. Goulaouic [1].

$$P = t^m D_t^m + P_1(t,x,D_x)t^{m-1}D_t^{m-1} + \cdots + P_m(t,x,D_x) \ , \tag{1.1}$$

$$P_j(t,x;D_x) = \sum_{|\alpha|\le j} a_{j,\alpha}(t,x) D_x^\alpha \qquad (1 \le j \le m) \ , \tag{1.2}$$

$$\operatorname{ord}_{D_x} P_j(0,x;D_x) \le 0 \qquad (1 \le j \le m) \ , \tag{1.3}$$

where m is a positive integer, and $D_t := \dfrac{\partial}{\partial t}$, $D_x := (D_{x_1}, \dots, D_{x_n})$, $D_{x_j} := \dfrac{\partial}{\partial x_j}$. Assume that the coefficients $a_{j,\alpha}$ ($|\alpha| \le j \le m$) are holomorphic in a neighborhood of $(t,x) = (0,0)$. M. Kashiwara and T. Oshima ([3], Definition 4.2) called such an operator "*an operator which has regular singularity in a weak sense along* $\Sigma_0 := \{ t = 0 \}$." For such operators, M. S. Baouendi and C. Goulaouic [1] showed fundamental theorems that are extensions of the Cauchy-Kowalevsky theorem and the Holmgren theorem (see Theorems 4.1, 4.2 given later).

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Remark 1.1. (1) M. S. Baouendi and C. Goulaouic considered Fuchsian partial differential equations with general weights. If P has weight ω, however, then $t^\omega P$ has weight 0, and hence the general case can be easily reduced to the case with weight 0 as for our problem.
(2) As for the spelling of the name of Kowalevskaya, I followed W. Walter [9], who pointed out that she wrote the paper [4] by the name of 'Sophie von Kowalevsky'.

We put $a_j(x) := P_j(0, x, D_x)$, and put

$$\begin{aligned}(1.4)\quad \mathcal{C}(x;\lambda) = \mathcal{C}[P](x;\lambda) &:= (\lambda)_m + a_1(x)(\lambda)_{m-1} \\ &\quad + \cdots + a_m(x) = t^{-\lambda}P(t^\lambda)|_{t=0}\ ,\end{aligned}$$

where $(\lambda)_k := \lambda(\lambda-1)\dots(\lambda-k+1)$. This polynomial in λ is called the *indicial polynomial* of P. A root of the equation $\mathcal{C}(x;\lambda) = 0$ in λ is called a *characteristic exponent* (or *characteristic index*) of P at x. We can not necessarily take a characteristic exponent $\lambda(x)$ as a holomorphic function of x, which is one of the difficulty to investigate Fuchsian partial differential equations in full generality.

In order to consider solutions in germ sense, put

$$\begin{aligned} B_R &:= \{\, x \in \boldsymbol{C}^n : |x| < R \,\}\ , \\ \mathcal{O}_0 &:= \operatorname*{indlim}_{R>0} \mathcal{O}(B_R)\ , \\ &\qquad \text{where } \mathcal{O}(\Omega) \text{ denotes the set of holomorphic functions on } \Omega\ , \\ \Delta_T &:= \{\, t \in \boldsymbol{C} : |t| < T \,\} \quad (T > 0)\ , \\ S_{\infty,T} &:= \mathcal{R}(\Delta_T \setminus \{0\}) \quad (\text{the universal covering of } \Delta_T \setminus \{0\})\ , \\ \widetilde{\mathcal{O}} &:= \operatorname*{indlim}_{T>0,R>0} \mathcal{O}(S_{\infty,T} \times B_R)\ . \end{aligned}$$

If the characteristic exponents $\{\,\lambda_l(0)\,\}_{l=1}^m$ of P at $x = 0$ do not differ by integer, that is, if $\lambda_l(0) - \lambda_{l'}(0) \notin \boldsymbol{Z} := \{\,\text{all the integers}\,\}$ for $l \neq l'$, then each characteristic exponent is simple at $x = 0$, and we can take the characteristic exponents $\lambda_l(x)$ $(1 \le l \le m)$ as holomorphic functions in a neighborhood of $x = 0$. In this case, every solution of $Pu = 0$ in $\widetilde{\mathcal{O}}$ can be represented as follows (see [6], [8]).

$$u(t,x) = \sum_{l=1}^{m} t^{\lambda_l(x)} \sum_{j=0}^{\infty} t^j u_{l,j}(t,x)\ , \tag{1.5}$$

$$u_{l,j}(t,x) = \sum_{k=1}^{r_{l,j}} u_{l,j,k}(x)(\log t)^{k-1}, \quad u_{l,j,k} \in \mathcal{O}_0\ , \tag{1.6}$$

where $r_{l,j}$ are positive integers and $r_{l,0} = 1$. Further, we can give $u_{l,0}(t,x) = u_{l,0,1}(x) \in \mathcal{O}_0$ arbitrarily, and then $u_{l,j}(t,x)$ $(j \ge 1)$ are determined uniquely.

Especially, we have a linear isomorphism

$$(1.7)\quad (\mathcal{O}_0)^m \ni (u_{l,0,1}(x))_{l=1}^m \overset{\sim}{\longmapsto} u(t,x) \in \operatorname{Ker}_{\widetilde{\mathcal{O}}} P := \{\, u \in \widetilde{\mathcal{O}} \colon Pu = 0 \,\} \ .$$

If some characteristic exponents differ by integer, then the situation seems far more complicated. One of the reasons is, of cource, that we can not necessarily take holomorphic characteristic exponents in general. For example, if $\mathcal{C}(x;\lambda) = \lambda^2 - x$, we need functions like $t^{\sqrt{x}} + t^{-\sqrt{x}}$, $\dfrac{t^{\sqrt{x}} - t^{-\sqrt{x}}}{\sqrt{x}}$, or derivatives of these.

The aim of this article is to construct a linear isomorphism $(\mathcal{O}_0)^m \overset{\sim}{\longrightarrow} \operatorname{Ker}_{\widetilde{\mathcal{O}}} P$ like (1.7) *without any assumptions on the characteristic exponents*, by essentially the same method as the method of Frobenius to ordinary differential equations with regular singularity at a point.

In the next section, we review the method of Frobenius, and modify it. In Section 3, we give the precise statement of our result. The first half of the main theorem is proved in Section 4. After giving some preliminary results in Sections 5, 6, and 7, we give a proof of the latter half of the main theorem in Section 8. Last, we give some variants of our result in Section 9.

Notation.

(i) The set of the integers is denoted by $\boldsymbol{Z}$, and the set of the nonnegative integers by $\boldsymbol{N}$.
(ii) If B is not open, $\mathcal{O}(B)$ denotes the set of the functions holomorphic on a neighborhood of B.
(iii) The closure of a set A is denoted by $\overline{A}$.
(iv) The set of the $N \times N$ matrices with complex entries is denoted by $M_N(\boldsymbol{C})$.
(v) The polynomial algebra in λ with the coefficients in a ring X is denoted by $X[\lambda]$.

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2. Review of the method of Frobenius to ordinary differential equations

In this section, we consider an ordinary differential equation with regular singularity at $t = 0 \in \boldsymbol{C}$.

$$(2.1)\qquad P = P(t, D_t) = t^m D_t^m + a_1(t) t^{m-1} D_t^{m-1} + \cdots + a_m(t) \ ,$$

$$a_j \in \mathcal{O}(\Delta_T) \quad (1 \le j \le m) \ ,$$

where m is a positive integer, $T > 0$, $D_t := \dfrac{d}{dt}$. This can be considered as the case $n = 0$.

The *indicial polynomial* of P is $\mathcal{C}(\lambda) = \mathcal{C}[P](\lambda) := (\lambda)_m + a_1(0)(\lambda)_{m-1} + \cdots + a_m(0) = t^{-\lambda}P(t^\lambda)|_{t=0}$, and a *characteristic exponent* is a root of the equation $\mathcal{C}(\lambda) = 0$.

Note that if we put $\vartheta := tD_t$, then we have $(\vartheta)_k = t^k D_t^k$ and $\mathcal{C}(\vartheta) = t^m D_t^m + a_1(0)t^{m-1}D_t^{m-1} + \cdots + a_m(0)$. Further if $F(\lambda) \in \boldsymbol{C}[\lambda]$, then we have $F(\vartheta) \circ (t^\lambda \times) = (t^\lambda \times) \circ F(\vartheta + \lambda)$, where $\circ$ denotes the composition (product) of two operators, and $t^\lambda \times$ is the multiplication operator by t^λ.

If the characteristic exponents $\{\, \lambda_l \,\}_{l=1}^m$ do not differ by integer, that is, if $\lambda_l - \lambda_{l'} \notin \boldsymbol{Z}$ for $l \ne l'$, then every solution of $Pu = 0$ in $\mathcal{O}(S_{\infty,T})$ is expanded as $u(t) = \displaystyle\sum_{l=1}^m t^{\lambda_l} \sum_{j=0}^\infty u_{l,j} t^j$, where $u_{l,j} \in \boldsymbol{C}$. Further, we can take arbitrary $u_{l,0}$ $(1 \le l \le m)$, and then $u_{l,j}$ $(j \ge 1)$ are determined uniquely by recursive equations of the form

$$\mathcal{C}(\lambda_l + j) \times u_{l,j} = [\text{Terms determined by } u_{l,0}, \ldots, u_{l,j-1}] \ .$$

Note that we have $\mathcal{C}(\lambda_l + j) \ne 0$ for $j \ge 1$ by the assumption that $\lambda_l + j$ $(j \in \boldsymbol{N} \setminus \{0\})$ is not a characteristic exponent. Especially, we have a base of $\operatorname{Ker}_{\mathcal{O}(S_{\infty,T})} P := \{\, u \in \mathcal{O}(S_{\infty,T}) \colon Pu = 0 \,\}$ in the form $\mathcal{B} = \{\, t^{\lambda_l} + \sum_{j=1}^\infty u_{l,j} t^{\lambda_l + j} \,\}_{l=1}^m$, and hence we have a linear isomorphism

$$\boldsymbol{C}^m \ni (u_{l,0})_{l=1}^m \overset{\sim}{\longmapsto} u(t) \in \operatorname{Ker}_{\mathcal{O}(S_{\infty,T})} P \ .$$

If some characteristic exponents do differ by integer, then there may appear some solutions with logarithmic terms in general. Namely, let μ_1, $\ldots$, μ_d be the distinct characteristic exponents, and let r_l be the multiplicity of μ_l $(l = 1, 2, \ldots, d)$ as a root of $\mathcal{C}(\lambda) = 0$. Then every solution of $Pu = 0$ in $\mathcal{O}(S_{\infty,T})$ is expanded as $u = \sum_{l=1}^d t^{\mu_l} \sum_{j=0}^\infty u_{l,j}(t) t^j$, $u_{l,j}(t) = \sum_{k=1}^{r_{l,j}} u_{l,j,k} (\log t)^{k-1}$, where $u_{l,j,k} \in \boldsymbol{C}$, $r_{l,j} \in \boldsymbol{N}$, and $r_{l,0} = r_l$.

The method of Frobenius, which is reviewed below, is a method to construct such solutions without being bothered by the tedious process to seek $u_{l,j,k}$ (not unique in general). Also in this case, we have a linear isomorphism

$$\boldsymbol{C}^m \ni (u_{l,0,k})_{1 \le l \le d; 1 \le k \le r_l} \ \overset{\sim}{\longmapsto} \ u(t) \in \operatorname{Ker}_{\mathcal{O}(S_{\infty,T})} P \ .$$

Let $\mu = \mu_l$ be a characteristic exponent with multiplicity $r = r_l$. From now on, we fix l and often omit the subscript l.

Let $0 = k_0 < k_1 < \cdots < k_q$ $(q = q_l)$ be the nonnegative integers k for which $\mu + k$ is also a characteristic exponent. Let $s_\delta = s_{l,\delta}$ be the multiplicity of $\mu + k_\delta$ $(\delta = 0, 1, \ldots, q)$ and put $R = R_l := \sum_{\delta=0}^q s_\delta$. Note that $s_0 = r$. Also note that if the characteristic exponents do not differ by integer, then $r_l = 1$, $q_l = 0$, and $R_l = 1$ for every l.

Take an integer $N \geq k_q$. If $q \geq 1$, then we can not solve, in general, the equation $Pu = 0$ in the form $u = t^\mu \sum_{j=0}^\infty u_j t^j$, since we have recursive equations $\mathcal{C}(\mu + j)u_j = [$ Terms determined by $u_0 - u_{j-1}]$, and there holds $\mathcal{C}(\mu + j) = 0$ for $j = k_\delta$ $(1 \leq \delta \leq q)$.

Considering $\lambda \in \boldsymbol{C}$ as a parameter moving near μ, we solve the equation

$$P(U) = \Big(\prod_{\nu=0}^{N} \mathcal{C}(\lambda + \nu)\Big) t^\lambda =: A(\lambda) t^\lambda \tag{2.2}$$

for U. Note that μ is a zero of $A(\lambda)$ of order R. We can obtain a formal solution U in the form

$$U = U(\lambda; t) = t^\lambda \widetilde{U}(\lambda; t) = t^\lambda \sum_{j=0}^\infty U_j(\lambda) t^j \ , \tag{2.3}$$

$$U_j \text{ is holomorphic in a } j\text{-independent neighborhood of } \mu \ , \tag{2.4}$$

$$\mu \text{ is a zero of } U_0 \text{ of order } R - r \ , \tag{2.5}$$

and we can show that $\widetilde{U}(\lambda; t)$ converges in $D \times \Delta_T$ for a neighborhood D of μ.

Since μ is a zero of A of order R, and since $U_0^{(R-r)}(\mu) \neq 0$, we can easily show that $\{ (\partial_\lambda^\nu U)(\mu; t) \colon R - r \leq \nu \leq R - 1 \}$ are independent solutions of $Pu = 0$. By constructing these solutions for each characteristic exponent $\mu = \mu_l$, we can construct a base of $\operatorname{Ker}_{\mathcal{O}(S_{\infty,T})} P$. This is the classical method of Frobenius (See, for example, [2] for a detail).

If we try to apply this method straightly to Fuchsian partial differential equations, we meet the following difficulties.

(A) Some characteristic exponents $\mu_l(x)$ may not be holomorphic in x, while we want solutions holomorphic in x.

(B) Even if $\mu_l(x)$ is holomorphic, the commutativity of two operators $(\partial_\lambda \cdots)|_{\lambda=\mu(x)}$ and P is no longer valid in general.

Of cource, the proof of the fact that the solution map $(\mathcal{O}_0)^m \longrightarrow \operatorname{Ker}_{\widetilde{\mathcal{O}}} P$ is an isomorphism would become far more difficult, while in the case of ordinary differential equations, it follows easily from the independence of the constructed solutions, since the solution space is finite dimensional.

Thus, we need to modify the classical method. The idea is very simple, once we notice it. We used a solution $U(\lambda; t)$ of

$$P(U) = \Big(\prod_{\nu=0}^{N} \mathcal{C}(\lambda + \nu)\Big) t^\lambda =: A(\lambda) t^\lambda \ , \tag{2.2}$$

which is holomorphic at $\lambda = \mu$, and obtained a solution $u = (\partial_\lambda^\nu U)(\mu; t)$ $(R - r \leq \nu \leq R - 1)$ of $Pu = 0$. This solution u is represented by Cauchy integral as

$$(\partial_\lambda^\nu U)(\mu;t) = \nu! \frac{1}{2\pi i} \int_\Gamma \frac{U(\lambda;t)}{(\lambda-\mu)^{\nu+1}} d\lambda \ , \tag{2.6}$$

where $\Gamma = \Gamma_l$ is a sufficiently small simple closed curve enclosing μ. If we consider a function

$$V(\lambda;t) := \nu! \frac{U(\lambda;t)}{(\lambda-\mu)^{\nu+1}} \ , \tag{2.7}$$

this is a solution of $P(V) = \nu! \dfrac{A(\lambda)}{(\lambda-\mu)^{\nu+1}} t^\lambda$, where the right hand side is also a polynomial in λ. Note that $V(\lambda;t)$ has a pole at $\lambda = \mu$ unlike U.

This consideration leads us to the following, where we write l.

Proposition 2.1. (1) *For every $F \in \boldsymbol{C}[\lambda]$, there exists a unique $\widetilde{V}[F] \in \mathcal{O}((\bigcup_{l=1}^d \Gamma_l) \times \Delta_T)$ such that $V = V[F](\lambda;t) := t^\lambda \widetilde{V}[F](\lambda;t)$ is a solution of the equation $P(V) = t^\lambda F(\lambda)$. Further, for every $l = 1, \dots, d$, the function*

$$u[l,F](t) := \frac{1}{2\pi i} \int_{\Gamma_l} V[F](\lambda;t)\, d\lambda \ \in \mathcal{O}(S_{\infty,T}) \tag{2.8}$$

is a solution of $Pu = 0$.
(2) *For $1 \le l \le d$ and $1 \le p \le r_l$, put $F_{l,p}(\lambda) := (\lambda - \mu_l)^{p-1}$ and put $u_{l,p} := u[l, F_{l,p}]$. Then, $\mathcal{B} := \{ u_{l,p} \colon 1 \le l \le d; 1 \le p \le r_l \}$ is a base of* $\operatorname{Ker}_{\mathcal{O}(S_{\infty,T})} P$. *Namely, we have a linear isomorphism*

$$\boldsymbol{C}^m \ni (\varphi_{l,p})_{\substack{1\le l\le d \\ 1\le p\le r_l}} \overset{\sim}{\longmapsto} u(t) = \sum_{l=1}^{d} \sum_{p=1}^{r_l} \varphi_{l,p} \cdot u_{l,p}(t) \in \operatorname{Ker}_{\mathcal{O}(S_{\infty,T})} P \ .$$

This modified version of the method of Frobenius can be applied to Fuchsian partial differential equations rather straightly, as seen in the next section.

3. The method of Frobenius to Fuchsian partial differential equations

Consider a Fuchsian partial differential operator P of the form (1.1)–(1.3). Assume that the coefficients $a_{j,\alpha}$ are holomorphic in a neighborhood of $\overline{\Delta_T} \times \overline{B_R}$ ($T > 0$, $R > 0$). Let μ_l ($l = 1, \dots, d$) be the distinct characteristic exponents of P at $x = 0$, and let r_l be the multiplicity of μ_l. By a technical reason, we take $\epsilon \ge 0$ such that $\operatorname{Re}\mu_l - \epsilon \notin \boldsymbol{Z}$ holds for all l. Take $L_l \in \boldsymbol{Z}$ such that $L_l + \epsilon < \operatorname{Re}\mu_l < L_l + \epsilon + 1$. Then, we have the following lemma.

Lemma 3.1. *For each $l = 1, \dots, d$, we can take a domain D_l in $\boldsymbol{C}$ enclosed by a simple closed curve Γ_l, such that the following holds.*

(a) $\mu_l \in D_l$ $(1 \le l \le d)$,
(b) $\overline{D_l} \cap \overline{D_{l'}} = \emptyset$ *if* $l \ne l'$,
(c) $\mathcal{C}(0; \lambda + j) \ne 0$ *for every* $\lambda \in \bigcup_{l=1}^d (\overline{D_l} \setminus \{ \mu_l \})$ *and every* $j \in \boldsymbol{N}$.

(d) $\overline{D_l} \subset \{ \lambda \in \boldsymbol{C} \colon L_l + \epsilon < \operatorname{Re} \lambda < L_l + \epsilon + 1 \}$ *for every* l.

Proof. We have only to take a sufficiently small D_l. Note that (c) is equivalent to

$$\overline{D_l} \cap \{ \mu_{l'} - j \in \boldsymbol{C} \colon 1 \leq l' \leq d, j \in \boldsymbol{N} \} = \{ \mu_l \} \text{ for every } l \tag{3.1}$$

and that $\{ \mu_{l'} - j \in \boldsymbol{C} \colon 1 \leq l' \leq d, j \in \boldsymbol{N} \}$ is a discrete set in $\boldsymbol{C}$. □

We also have the following lemma.

Lemma 3.2. *For each* $l = 1, \ldots d$, *take* D_l *and* Γ_l *as in Lemma* 3.1. *There exists* $R_0 > 0$ *and monic polynomials* $E_l(x; \lambda) \in \mathcal{O}(B_{R_0})[\lambda]$ $(1 \leq l \leq d)$ *such that the following holds.*

(e) $\mathcal{C}(x; \lambda) = \prod_{l=1}^{d} E_l(x; \lambda)$ *and* $E_l(0; \lambda) = (\lambda - \mu_l)^{r_l}$ $(1 \leq l \leq d)$.
(f) *For every* l, *if* $E_l(x; \lambda) = 0$ *and* $x \in B_{R_0}$, *then* $\lambda \in D_l$.
(g) $\mathcal{C}(x; \lambda + j) \neq 0$ *for all* $x \in B_{R_0}$, *all* $\lambda \in \bigcup_{l=1}^{d} \Gamma_l$, *and all* $j \in \boldsymbol{N}$.

Proof. By (c) in Lemma 3.1, (g) holds for $x = 0$. Hence, if we take a sufficiently small $R_0 > 0$, then (g) holds. This condition (g) implies that the number of the roots in D_l of $\mathcal{C}(x; \lambda) = 0$ is constant r_l for every $x \in B_{R_0}$. Let $\lambda_j(x)$ $(1 \leq j \leq r_l)$ be the roots of $\mathcal{C}(x; \lambda) = 0$ in D_l. Then, we have

$$\sum_{j=1}^{r_l} \lambda_j(x)^k = \frac{1}{2\pi i} \int_{\Gamma_l} \frac{\partial_\lambda \mathcal{C}(x; \lambda)}{\mathcal{C}(x; \lambda)} \lambda^k \, d\lambda \ \in \mathcal{O}(B_{R_0}) \ .$$

Since the fundamental symmetric forms of $(\lambda_j(x))_{j=1}^{r_l}$ are polynomials of $\sum_{j=1}^{r_l} \lambda_j(x)^k$ $(1 \leq k \leq r_l)$ with the constant coefficients, the polynomial $E_l(x; \lambda) := \prod_{j=1}^{r_l} (\lambda - \lambda_j(x))$ has the coefficients in $\mathcal{O}(B_{R_0})$. Thus, we have (e).

Finally, (f) follows easily from (e) and (g) for $j = 0$. □

From now on in this article, we fix Γ_l, B_{R_0} and others given in these two lemmata. The following is the main result, which we call *the method of Frobenius to Fuchsian partial differential equations.*

Theorem 3.3. (1) *For every* $R' \in (0, R_0)$ *and for every* $F \in \mathcal{O}(B_{R'})[\lambda]$, *there exists a unique* $\widetilde{V}[F] \in \mathcal{O}(\{t = 0\} \times B_{R'} \times (\bigcup_{l=1}^{d} \Gamma_l))$ *such that* $V = V[F](t, x; \lambda) := t^\lambda \widetilde{V}[F](t, x; \lambda)$ *is a solution of the equation*

$$P(V) = t^\lambda F(x; \lambda) \ . \tag{3.2}$$

For every $l = 1, \ldots, d$, *the function*

$$u[l, F](t, x) := \frac{1}{2\pi i} \int_{\Gamma_l} V[F](t, x; \lambda) \, d\lambda \tag{3.3}$$

is a solution of $Pu = 0$. *Further, for every* $R'' \in (0, R')$, *there exists* $T'' > 0$ *such that* $u[l, F] \in \mathcal{O}(S_{\infty, T''} \times B_{R''})$.

(2) *For* $1 \le l \le d$ *and* $1 \le p \le r_l$, *put* $F_{l,p}(x;\lambda) := S_l(x;\lambda)\partial_\lambda^p E_l(x;\lambda)$, *where* $S_l(x;\lambda) := \mathcal{C}(x;\lambda)/E_l(x;\lambda) \in \mathcal{O}(B_{R_0})[\lambda]$. *For* $\varphi \in \mathcal{O}_0$, *we put* $u_{l,p}[\varphi](t,x) := u[l, \varphi \cdot F_{l,p}](t,x)$. *Then, we have a linear isomorphism*

$$(\mathcal{O}_0)^m \ni (\varphi_{l,p})_{\substack{1\le l\le d,\\ 1\le p\le r_l}} \overset{\sim}{\longmapsto} \sum_{l=1}^{d}\sum_{p=1}^{r_l} u_{l,p}[\varphi_{l,p}] \in \mathrm{Ker}_{\widetilde{\mathcal{O}}} P. \tag{3.4}$$

Remark 3.4. (1) If the characteristic exponents do not differ by integer, then our solution map (3.4) is just the same as (1.7).
(2) As a matter of fact, we shall prove a little stronger result than (2) of the theorem. For $\theta \in (0,\infty]$, $T' > 0$, and $R' > 0$, put

$$S_{\theta,T'} := \{\, t \in S_{\infty,T'} : |\arg t| < \theta \,\} \ , \tag{3.5}$$

$$\widetilde{\mathcal{O}_{\theta,R'}} := \bigcap_{0<R''<R'} \bigcup_{T''>0} \mathcal{O}(S_{\theta,T''} \times B_{R''}) \tag{3.6}$$

$$= \{\, \phi \colon \text{for every } R'' \in (0,R'), \text{ there exists } T'' > 0 \text{ such that } \phi \in \mathcal{O}(S_{\theta,T''} \times B_{R''}) \,\} \ . \tag{3.7}$$

By this notation, the last part of Theorem 3.3-(1) says that $u[l,F] \in \widetilde{\mathcal{O}_{\infty,R'}}$. We shall show that for every $R' \in (0,R_0)$ and every $\theta \in (0,\infty]$, our solution map (3.4) induces a linear isomorphism

$$(\mathcal{O}(B_{R'}))^m \overset{\sim}{\longrightarrow} \mathrm{Ker}_{\widetilde{\mathcal{O}_{\theta,R'}}} P \ . \tag{3.8}$$

As a result of this, $\mathrm{Ker}_{\widetilde{\mathcal{O}_{\theta,R'}}} P$ do not depend on $\theta \in (0,\infty]$, which implies that if $\theta > 0$, $T' \in (0,T)$, $R' \in (0,R_0)$, and if $u \in \mathcal{O}(S_{\theta,T'} \times B_{R'})$ satisfies $Pu = 0$, then for every $R'' \in (0,R')$, there exists $T'' > 0$ such that u can be extended to $\mathcal{O}(S_{\infty,T''} \times B_{R''})$.

We end this section by giving an asymptotic expansion of the solution $u_{l,p}[\varphi]$.

Since $\widetilde{V}[F] \in \mathcal{O}(\{t=0\} \times B_{R'} \times (\bigcup_{l=1}^d \Gamma_l))$, we can expand this function with respect to t as

$$\widetilde{V}[F](t,x;\lambda) = \sum_{j=0}^{\infty} t^j \widetilde{V}_j[F](x;\lambda), \quad \widetilde{V}_j[F] \in \mathcal{O}(B_{R'} \times (\bigcup_{l=1}^d \Gamma_l)) \ . \tag{3.9}$$

We can also expand the operator P as

$$P = \mathcal{C}(x;\vartheta) + \sum_{h=1}^{\infty} t^h Q_h(x;\vartheta,D_x) \ , \tag{3.10}$$

where $Q_h(x,\lambda,\xi) \in \mathcal{O}(\overline{B_R})[\lambda,\xi]$. Hence, from $P(t^\lambda \widetilde{V}[F]) = t^\lambda F(x;\lambda)$, we have a system of equations

$$\mathcal{C}(x;\lambda)\widetilde{V}_0[F](x;\lambda) = F(x;\lambda) \ , \tag{3.11}$$

$$\mathcal{C}(x;\lambda+j)\widetilde{V}_j[F](x;\lambda) = -\sum_{h=1}^{j} Q_h(x;\lambda+j-h,D_x)\widetilde{V}_{j-h}[F](x;\lambda) \quad (j \geq 1) \ . \tag{3.12}$$

Hence, we have

$$\widetilde{V}_0[F] = \frac{F}{\mathcal{C}} \ , \tag{3.13}$$

$$\widetilde{V}_j[F] \in \frac{1}{\prod_{\nu=1}^{j}\mathcal{C}(x;\lambda+\nu)^{m_{j,\nu}}} \times \mathcal{O}(B_{R'})[\lambda] \ , \tag{3.14}$$

for some $m_{j,\nu} \in \boldsymbol{N}$.

According to the expansion of $\widetilde{V}[\varphi \cdot F_{l,p}]$, we have an asymptotic expansion of $u_{j,p}[\varphi](t,x)$ as

$$u_{l,p}[\varphi](t,x) = \sum_{j=0}^{\infty} u_{l,p,j}[\varphi](t,x)t^j \ , \tag{3.15}$$

$$u_{l,p,j}[\varphi](t,x) = \frac{1}{2\pi i}\int_{\Gamma_l} \widetilde{V}_j[\varphi \cdot F_{l,p}](x;\lambda)t^{\lambda}\, d\lambda \ \in \mathcal{O}(S_{\infty,\infty} \times B_{R'}) \ . \tag{3.16}$$

Hence, for every fixed $x = x_0$, the solution $u_{l,p}[\varphi](t,x_0)$ has the form of

$$\sum_{\nu=1}^{d'_l}\sum_{j=0}^{\infty}\sum_{k=1}^{r'_{\nu,j}} v_{\nu,j,k}t^{\mu'_{l,\nu}+j}(\log t)^{k-1}$$

for some $v_{\nu,j,k} = v_{l,p,\nu,j,k} \in \boldsymbol{C}$, some $r'_{\nu,j} = r'_{l,p,\nu,j} \in \boldsymbol{N} \setminus \{0\}$, and some $\mu'_{l,\nu} \in D_l$. Especially, since

$$u_{l,p,0}[\varphi](t,x) = \frac{1}{2\pi i}\int_{\Gamma_l} \varphi(x)\frac{\partial_\lambda^p E_l(x;\lambda)}{E_l(x;\lambda)}t^{\lambda}\, d\lambda \ ,$$

and since $d'_l = 1$ and $\mu'_{l,1} = \mu_l$ at $x = 0$, we can expand $u_{l,p}[\varphi]$ at $x = 0$ as

$$u_{l,p}[\varphi](t,0) = \varphi(0)\frac{(r_l)_p}{(p-1)!}t^{\mu_l}(\log t)^{p-1} + \sum_{j=1}^{\infty}\sum_{k=1}^{r'_{l,p,j}} v_{l,p,j,k}t^{\mu_l+j}(\log t)^{k-1} \ , \tag{3.17}$$

for some $v_{l,p,j,k} \in \boldsymbol{C}$.

Remark 3.5. If the indicial polynomial $\mathcal{C}(x;\lambda)$ of P is independent of x, then even if some characteristic exponents do differ by integer, we have an expansion of the form

$$u_{l,p}[\varphi](t,x) = \sum_{j=0}^{\infty}\sum_{k=1}^{r_{l,p,j}} u_{l,p,j,k}(x)t^{\mu_l+j}(\log t)^{k-1}, \quad u_{l,p,j,k} \in \mathcal{O}_0$$

as in the case of ordinary differential equations, though $r_{l,p,j}$ may diverge to ∞ as $j \to \infty$, while $r_{l,p,j} \leq m$ in the case of ordinary differential equations.

4. Proof of Theorem 3.3-(1)

In this section, we prove Theorem 3.3-(1). Let P be a Fuchsian partial differential operator considered in Section 3. Namely, P is an operator of the form (1.1)–(1.3), and the coefficients are holomorphic in a neighborhood of $\overline{\Delta_T} \times \overline{B_R}$ $(T > 0,\ R > 0)$.

First, we give two fundamental results of M. S. Baouendi and C. Goulaouic [1] in a form of later convenience. The first one is the unique solvability of the equation in the category of holomorphic functions, which corresponds to the Cauchy-Kowalevsky theorem.

Theorem 4.1. *Assume the condition*

$$\mathcal{C}(0; j) \neq 0 \text{ for every } j \in \boldsymbol{N}\ .$$

Then, there exist $T' > 0$ and $R' > 0$ such that for every $f \in \mathcal{O}(\Delta_T \times B_R)$, there exists a unique $u \in \mathcal{O}(\Delta_{T'} \times B_{R'})$ such that $Pu = f$.

The second one is the uniqueness of solutions in a wider class, which corresponds to the Holmgren theorem. We consider solutions in real domain.

Theorem 4.2. *There exists $L \in \boldsymbol{N}$ such that if $u \in C^L([0, T'] \times U)$, where $T' > 0$ and U is an open neighborhood of $x = 0$ in $\boldsymbol{R}^n$, if $D_t^j u|_{t=0} = 0$ $(0 \le j \le L-1)$, and if $Pu = 0$ on $(0, T') \times U$, then there exists $T'' > 0$ and an open neighborhood U' of $x = 0$ such that $u = 0$ in $[0, T''] \times U'$.*

Now, we can prove Theorem 3.3-(1) by using Theorem 4.1. (Theorem 4.2 is used in Section 8.)

Proof of Theorem 3.3-(1). Suppose that $0 < R' < R_0$ and $F \in \mathcal{O}(B_{R'})[\zeta]$. First, note that P can be written as

$$P = \mathcal{P}(t, x; \vartheta, D_x)\ ,$$

where $\mathcal{P}(t, x; \lambda, \xi)$ is a polynomial in (λ, ξ) with the coefficients in $\mathcal{O}(\overline{\Delta_T} \times \overline{B_R})$. The equation $P(t^\zeta \widetilde{V}(t, x; \zeta)) = t^\zeta F(x; \zeta)$ is equivalent to $\mathcal{P}(t, x; \vartheta + \zeta, D_x)\widetilde{V} = F(x; \zeta)$, and the operator $Q := \mathcal{P}(t, x; \vartheta+\zeta, D_x)$ is also a Fuchsian partial differential operator of the variables (t, x, ζ). The indicial polynomial of Q is $\mathcal{C}[Q](x, \zeta; \lambda) = \mathcal{C}[P](x; \lambda + \zeta)$.

If we fix $(x, \zeta) = (x_0, \zeta_0) \in B_{R'} \times \left(\bigcup_{l=1}^d \Gamma_l\right)$, then $\mathcal{C}[Q](x_0, \zeta_0; j) \neq 0$ for every $j \in \boldsymbol{N}$, by (g) of Lemma 3.2. Hence, we can use Theorem 4.1 for Q and (x_0, ζ_0) instead of P and 0, and hence there exist $T' > 0$, $R'' > 0$ and a unique $\widetilde{V}(t, x; \zeta) \in \mathcal{O}(\Delta_{T'} \times B_{R''}(x_0, \zeta_0))$ such that $Q(\widetilde{V}) = F(x; \zeta)$, where $B_{R''}(x_0, \zeta_0)$ is the open ball in $\boldsymbol{C}^{n+1}$ centered at (x_0, ζ_0) with the radius R''. Since (x_0, ζ_0) is an arbitrary point of $B_{R'} \times \left(\bigcup_{l=1}^d \Gamma_l\right)$, and since $\widetilde{V}$ is unique, there exists an open neighborhood Ω of $\{t = 0\} \times B_{R'} \times \left(\bigcup_{l=1}^d \Gamma_l\right)$ in $\boldsymbol{C} \times \boldsymbol{C}^n \times \boldsymbol{C}$ such that $\widetilde{V} \in \mathcal{O}(\Omega)$. This means that for every $R'' \in (0, R')$, there exists $T'' > 0$ such that $\widetilde{V} \in \mathcal{O}(\Delta_{T''} \times B_{R''} \times \left(\bigcup_{l=1}^d \Gamma_l\right))$. From this, it is almost trivial that $u[l, F] \in \widetilde{\mathcal{O}_{\infty, R'}}$.

Since $\displaystyle\int_{\Gamma_l} F(x;\zeta)t^{\zeta}\,d\zeta = 0$, it is also trivial that $u[l,F]$ is a solution of $Pu = 0$. □

5. Function spaces measuring the order of functions

In this section, we introduce some function spaces, which "measure" the order of functions as $t \to 0$.

Definition 5.1. For $\theta \in (0,\infty]$, $T > 0$, and $R > 0$, put

$$W(\theta,T,R) := \{\, \phi \in \mathcal{O}(S_{\theta,T} \times B_R) : \text{for every } \theta' \in (0,\theta) \text{ and every } R' \in (0,R), \text{ there holds } \sup_{|x|\le R'} |\phi(t,x)| \to 0 \text{ (as } t \to 0 \text{ in } S_{\theta',T}) \,\} \ , \tag{5.1}$$

where $S_{\theta,T}$ is defined by (3.5), and put

$$\widetilde{W}(\theta,R) := \{\, \phi \in \widetilde{\mathcal{O}_{\theta,R}} : \text{for every } R' \in (0,R), \text{ there exists } T' > 0 \text{ such that } \phi \in W(\theta,T',R') \,\} \ . \tag{5.2}$$

Further, for $a \in \boldsymbol{R}$, put

$$W^{(a)}(\theta,T,R) := t^a \times W(\theta,T,R) \ , \tag{5.3}$$

$$\widetilde{W}^{(a)}(\theta,R) := t^a \times \widetilde{W}(\theta,R) \ . \tag{5.4}$$

We have the following fundamental properties of these function spaces.

Lemma 5.2. (1) *$\vartheta^l(W(\theta,T,R)) \subset W(\theta,T,R)$ for all $l \in \boldsymbol{N}$. Namely, if $\phi \in W(\theta,T,R)$, then $\vartheta^l\phi \in W(\theta,T,R)$ for all $l \in \boldsymbol{N}$.*
(2) *$D_{x_j}(W(\theta,T,R)) \subset W(\theta,T,R)$ $(1 \le j \le n)$.*
(3) *$\widetilde{W}^{(a)}(\theta,R) = \{\, \phi :$ for every $R' \in (0,R)$, there exists $T' > 0$ such that $\phi \in W^{(a)}(\theta,T',R') \,\}$.*
(4) *If $a' < a$, then $W^{(a)}(\theta,T,R) \subset W^{(a')}(\theta,T,R)$ and $\widetilde{W}^{(a)}(\theta,R) \subset \widetilde{W}^{(a')}(\theta,R)$.*
(5) *$t \times W^{(a)}(\theta,T,R) \subset W^{(a+1)}(\theta,T,R)$, $t \times \widetilde{W}^{(a)}(\theta,R) \subset \widetilde{W}^{(a+1)}(\theta,R)$.*
(6) *$\partial_t(W^{(a)}(\theta,T,R)) \subset W^{(a-1)}(\theta,T,R)$, $\partial_t(\widetilde{W}^{(a)}(\theta,R)) \subset \widetilde{W}^{(a-1)}(\theta,R)$.*
(7) *If $B(t,x;D_x)$ is a differential operator of x with the coefficients in $\mathcal{O}(\Delta_T \times B_R)$, then*

$$B(t,x;D_x)(W^{(a)}(\theta,T,R)) \subset W^{(a)}(\theta,T,R) \ , \tag{5.5}$$

$$B(t,x;D_x)(\widetilde{W}^{(a)}(\theta,R)) \subset \widetilde{W}^{(a)}(\theta,R) \ . \tag{5.6}$$

Proof. We show only (1) and (2). The rest is straightforward and easy after these two are proved.

(1) We have only to show the case $l = 1$. Suppose that $\phi \in W(\theta,T,R)$ and $\theta' \in (0,\theta)$. We take $\theta'' \in (\theta',\theta)$. Then, there exists $\delta > 0$ such that

if $t \in S_{\theta',\infty}$, $\rho \in \boldsymbol{C}$, and $|\rho| \le \delta|t|$, then $t+\rho \in S_{\theta'',\infty}$. Take the circle Γ_t centered at t with the radius $\delta|t|$. We can write

$$D_t\phi(t,x) = \frac{1}{2\pi i}\int_{\Gamma_t} \frac{\phi(\tau,x)}{(\tau-t)^2}\,d\tau \ ,$$

for $t \in S_{\theta',T/(1+\delta)}$. From this, we have

$$|\vartheta\phi(t,x)| \le \frac{1}{\delta}\sup_{|\tau|\le(1+\delta)|t|;|\arg\tau|\le\theta''}|\phi(\tau,x)|$$

for $t \in S_{\theta',T/(1+\delta)}$ and $x \in B_R$. Thus, $\vartheta\phi \in W(\theta,T,R)$.

(2) Suppose that $\phi \in W(\theta,T,R)$ and that $R' \in (0,R)$. Take $R'' \in (R',R)$. Similarly to the proof of (1), we can show

$$\sup_{|x|\le R'}|D_{x_j}\phi(t,x)| \le \frac{1}{R''-R'}\sup_{|\zeta|\le R''}|\phi(t,\zeta)|$$

for $t \in S_{\theta,T}$. □

Proposition 5.3. *For a simple closed curve Γ in $\boldsymbol{C}$ and a function $\widetilde{V}(t,x;\lambda) \in \mathcal{O}(\Delta_{T'} \times B_{R'} \times \Gamma)$, put*

$$u(t,x) := \int_\Gamma \widetilde{V}(t,x;\lambda)t^\lambda\,d\lambda \ \in \mathcal{O}(S_{\infty,T'} \times B_{R'}) \ .$$

If $a < \min\{\operatorname{Re}\lambda : \lambda \in \Gamma\}$, then we have

$$u \in W^{(a)}(\infty,T',R') \ .$$

If $\widetilde{V} \in \mathcal{O}(\{t=0\} \times B_{R'} \times \Gamma)$ instead, then we have $u \in \widetilde{W}^{(a)}(\infty,R')$.

Proof. Suppose that $\widetilde{V} \in \mathcal{O}(\Delta_{T'} \times B_{R'} \times \Gamma)$. Since

$$t^{-a}u(t,x) = \int_\Gamma \widetilde{V}(t,x;\lambda)t^{\lambda-a}\,d\lambda \ ,$$

we can easily show that $t^{-a}u \in W(\infty,T',R')$, that is, $u \in W^{(a)}(\infty,T',R')$. The last part is now easy. □

6. Euler equations with holomorphic parameters

In order to prove the main theorem, we need some results on the equation $E_l(x;\vartheta)u = f(t,x)$ and $\mathcal{C}(x;\vartheta)u = f(t,x)$. These are ordinary differential equations of special type, so-called Euler equations, with holomorphic parameter x. In this section, we study about equations of this type.

First, we consider a "good" base of the solution space of the homogeneous equation $E(x;\vartheta)u = 0$, where $E(x;\lambda) \in \mathcal{O}(B_R)[\lambda]$ $(R > 0)$ is a monic polynomial of degree r.

Definition 6.1. For $p = 1, 2, \dots, r$, put

$$w_p(t,x) := \frac{1}{2\pi i}\int_\Gamma \frac{\partial_\lambda^p E(x;\lambda)}{E(x;\lambda)} t^\lambda \, d\lambda \quad \in \mathcal{O}(S_{\infty,\infty} \times B_R) \ , \tag{6.1}$$

where $\Gamma = \Gamma_x$ is a simple closed curve (or a sum of disjoint simple closed curves) in $\boldsymbol{C}$ enclosing all the roots λ of $E(x;\lambda) = 0$. Note that w_p is independent of the choice of such Γ.

Example 6.2. If $E(x;\lambda) = \lambda^2 - x$, then

$$w_1 = \frac{1}{2\pi i}\int_\Gamma \frac{2\lambda}{\lambda^2 - x} t^\lambda \, d\lambda = t^{\sqrt{x}} + t^{-\sqrt{x}} \ ,$$

$$w_2 = \frac{1}{2\pi i}\int_\Gamma \frac{2}{\lambda^2 - x} t^\lambda \, d\lambda = \frac{t^{\sqrt{x}} - t^{-\sqrt{x}}}{\sqrt{x}} \ ,$$

where Γ encloses $\pm\sqrt{x}$. In general, w_p is a "symmetrization of (higher) divided differences" of $\{ t^{\lambda_j(x)} \}_{j=1}^r$, where $\{ \lambda_j(x) \}_{j=1}^r$ are the roots of $E(x;\lambda) = 0$. S. Wakabayashi told the author that divided differences are represented in a simple way by Cauchy integrals, which was a great advice for the author. In this article, we do not use any higher divided differences explicitly, and hence we omit the detail.

Let $T > 0$ and $\theta \in (0, \infty]$.

Proposition 6.3. (1) *For every fixed $x \in B_R$, the set of functions $\mathcal{B} := \{ w_p(\cdot, x) \}_{1 \le p \le r}$ is a base of*

$$\operatorname{Ker}_{\mathcal{O}(S_{\theta,T})} E(x;\vartheta) := \{ u \in \mathcal{O}(S_{\theta,T}) \colon E(x;\vartheta)u = 0 \} \ .$$

(2) *If $u \in \mathcal{O}(S_{\theta,T} \times B_R)$ satisfies $E(x;\vartheta)u = 0$, then there exists unique $\varphi_p \in \mathcal{O}(B_R)$ $(1 \le p \le r)$ such that $u(t,x) = \sum_{p=1}^r \varphi_p(x) w_p(t,x)$.*

To prove this proposition, we use the following lemma, whose proof is easy and omitted.

Lemma 6.4. *Let $F \in \boldsymbol{C}[\lambda]$ be a monic polynomial of degree $r \in \boldsymbol{N} \setminus \{0\}$. If Γ is a simple closed curve enclosing all the roots of $F(\lambda) = 0$, then*

$$\int_\Gamma \frac{\lambda^\nu}{F(\lambda)} \, d\lambda = 0 \quad (0 \le \nu < r-1,\ \nu \in \boldsymbol{N}) \ ,$$

$$\int_\Gamma \frac{\lambda^{r-1}}{F(\lambda)} \, d\lambda = 2\pi i \ .$$

Proof of Proposition 6.3. (1) We fix $x \in B_R$. Since $E(x;\vartheta)t^\lambda = E(x;\lambda)t^\lambda$, $u = w_p(\cdot, x)$ is a solution of $E(x;\vartheta)u = 0$.

First, we consider $w_p(\cdot, x)$ as elements of $\mathcal{O}(S_{\theta,\infty})$. By Lemma 6.4 we have

$$(\vartheta^\nu w_p)(1,x) = 0 \ (0 \le \nu < p-1) \ \text{ and } \ (\vartheta^{p-1} w_p)(1,x) = (r)_p \ne 0 \ . \tag{6.2}$$

Using these, we can easily show that $\mathcal{B}$ is linearly independent in $\mathcal{O}(S_{\theta,\infty})$, and hence in $\mathcal{O}(S_{\theta,T})$. Since $\dim \operatorname{Ker}_{\mathcal{O}(S_{\theta,T})} E(x;\vartheta) = r$, the set $\mathcal{B}$ is a base.

(2) By (1), $\varphi_p(x)$ is uniquely determined for each x. The problem is the holomorphy in x. By the standard theory of ordinary differential equations, u can be extended to $\mathcal{O}(S_{\theta,\infty} \times B_R)$ (as a matter of fact, to $\mathcal{O}(S_{\infty,\infty} \times B_R)$). By (6.2), we have

$$(\vartheta^\nu u)(1,x) = (r)_{\nu+1}\, \varphi_{\nu+1}(x) + \sum_{p=1}^{\nu} \varphi_p(x)(\vartheta^\nu w_p)(1,x) \quad (0 \le \nu < r) \ .$$

From this, we can easily show that $\varphi_p \in \mathcal{O}(B_R)$ for $p = 1, 2, \ldots, r$. □

Next, we consider non-homogeneous equations.

Lemma 6.5. *Suppose that $g \in \mathcal{O}(S_{\theta,T})$ satisfy the condition*

(B) $g(t) \to 0$ *as* $t \to 0$ *in* $S_{\theta,T}$.

(1) *Fix* $T_1 \in (0,T)$. *If* $\operatorname{Re}\lambda > 0$, *then*

$$v[\lambda; g](t) = t^\lambda \int_{T_1}^{t} \tau^{-\lambda-1} g(\tau)\, d\tau$$

is a solution of $(\vartheta - \lambda)v = g(t)$, *and* $v[\lambda; g](t)$ *is holomorphic in* (λ, t) *on* $\boldsymbol{C}_+ \times S_{\theta,T}$, *where* $\boldsymbol{C}_\pm := \{\, \lambda \in \boldsymbol{C} : \pm \operatorname{Re}\lambda > 0 \,\}$. *Further,* $v[\lambda; g](t)$ *satisfies the following estimate for every* $\theta' \in (0,\theta)$ *and every* $M > 0$.

$$\text{(6.3)} \qquad \sup_{|t| \le \rho;\, |\arg t| \le \theta'} |v[\lambda; g](t)| \le \frac{C}{\operatorname{Re}\lambda} \Big\{ \rho^{\operatorname{Re}\lambda/2} \sup_{|\tau| \le T_1;\, |\arg\tau| \le \theta'} |g(\tau)| + \sup_{|\tau| \le \sqrt{\rho};\, |\arg\tau| \le \theta'} |g(\tau)| \Big\} \ ,$$
$$\textit{for every } \rho \in (0, T_1) \textit{ and every } \lambda \in \boldsymbol{C}_+ \cap B_M \ ,$$

where $C = C_{\theta', T_1, M}$ *is independent of* ρ, λ *and* g.

(2) *If* $\operatorname{Re}\lambda < 0$, *then*

$$v[\lambda; g](t) = t^\lambda \int_0^t \tau^{-\lambda-1} g(\tau)\, d\tau = \int_0^1 \sigma^{-\lambda-1} g(\sigma t)\, d\sigma$$

is a solution of $(\vartheta - \lambda)v = g(t)$, *and* $v[\lambda; g](t)$ *is holomorphic in* (λ, t) *on* $\boldsymbol{C}_- \times S_{\theta,T}$. *Further,* $v[\lambda; g](t)$ *satisfies the following estimate for every* $\theta' \in (0,\theta)$, *every* $\rho \in (0,T)$, *and every* $\lambda \in \boldsymbol{C}_-$.

$$\text{(6.4)} \qquad \sup_{|t| \le \rho;\, |\arg t| \le \theta'} |v[\lambda; g](t)| \le \frac{1}{|\operatorname{Re}\lambda|} \sup_{|\tau| \le \rho;\, |\arg\tau| \le \theta'} |g(\tau)| \ .$$

Note that $v[\lambda; g] \to 0$ as $t \to 0$ in $S_{\theta,T}$ in both cases. Also note that $v[\lambda; g]$ depends on the choice of T_1 in the case (1).

Proof. We show only the estimate in (1). The rest is easy.

We may assume that $\rho \le \min\{1, T_1^2\}$. By putting $\alpha := \arg t$, we have

$$v[\lambda; g](t) = t^\lambda \int_{T_1}^{T_1 e^{i\alpha}} \tau^{-\lambda-1} g(\tau)\, d\tau + t^\lambda \int_{T_1 e^{i\alpha}}^{\sqrt{|t|} e^{i\alpha}} \tau^{-\lambda-1} g(\tau)\, d\tau$$

$$+ t^{\lambda} \int_{\sqrt{|t|}e^{i\alpha}}^{|t|e^{i\alpha}} \tau^{-\lambda-1} g(\tau)\, d\tau =: I_1 + I_2 + I_3 \ ,$$

and we have for $|t| \le \rho \le T_1$,

$$\begin{aligned}
|I_1| &\le C|t|^{\operatorname{Re}\lambda} \sup_{|\tau|=T_1;|\arg\tau|\le\theta'} |g(\tau)| \ , \\
|I_2| &\le C|t|^{\operatorname{Re}\lambda} \int_{\sqrt{|t|}}^{T_1} \sigma^{-\operatorname{Re}\lambda-1} |g(\sigma e^{i\alpha})|\, d\sigma \\
&\le C|t|^{\operatorname{Re}\lambda} \Big(\sup_{|\tau|\le T_1;|\arg\tau|\le\theta'} |g(\tau)| \Big) \int_{\sqrt{|t|}}^{T_1} \sigma^{-\operatorname{Re}\lambda-1}\, d\sigma \\
&\le C\frac{1}{\operatorname{Re}\lambda} |t|^{\operatorname{Re}\lambda/2} \sup_{|\tau|\le T_1;|\arg\tau|\le\theta'} |g(\tau)| \ , \\
|I_3| &\le C\frac{1}{\operatorname{Re}\lambda} \sup_{|\tau|\le\sqrt{|t|};|\arg\tau|\le\theta'} |g(\tau)| \ ,
\end{aligned}$$

where C denotes constants that may be different in each appearance, but independent of ρ, λ with $|\lambda| \le M$, and g. The estimate (6.3) follows easily from these estimates. □

Lemma 6.6. *Suppose that $g \in \mathcal{O}(S_{\theta,T})$ satisfies the condition* (B) *in Lemma* 6.5. *Let $v[\lambda; g]$ be the solution given in Lemma* 6.5. *Let $\lambda_1, \dots, \lambda_r \in \boldsymbol{C}$ satisfy that all $\operatorname{Re}\lambda_\nu$ $(1 \le \nu \le r)$ has the same sign. Put*

$$v[\lambda_1, \dots, \lambda_r; g](t) := \frac{1}{2\pi i} \int_\Gamma \frac{1}{(\zeta-\lambda_1)\dots(\zeta-\lambda_r)} v[\zeta; g](t)\, d\zeta \quad \in \mathcal{O}(S_{\theta,T}) \ ,$$

where Γ is a simple closed curve in $\boldsymbol{C}$ enclosing all λ_ν and being disjoint with the imaginary axis. Then, $v = v[\lambda_1, \dots, \lambda_r; g]$ is a solution of the equation

$$(\vartheta - \lambda_1)\dots(\vartheta-\lambda_r)v = g(t) \ .$$

Further, $v[\lambda_1, \dots, \lambda_r; g](t) \to 0$ as $t \to 0$ in $S_{\theta,T}$.

Proof. Since $(\vartheta - \lambda_r)v[\zeta; g](t) = (\zeta - \lambda_r)v[\zeta; g](t) + g(t)$, if $r \ge 2$, then we have

$$(\vartheta - \lambda_r)v[\lambda_1, \dots, \lambda_r; g](t) = v[\lambda_1, \dots, \lambda_{r-1}; g](t) \ ,$$

by Lemma 6.4. Hence, by the induction on r, $v[\lambda_1, \dots, \lambda_r; g]$ is a solution of the equation $(\vartheta-\lambda_1)\dots(\vartheta-\lambda_r)v = g(t)$. From the estimate of $v[\zeta; g](t)$ given in Lemma 6.5, we can easily show that $v[\lambda_1, \dots, \lambda_r; g](t) \to 0$ as $t \to 0$ in $S_{\theta,T}$. □

Proposition 6.7. *Let $F(x; \lambda) \in \mathcal{O}(B_R)[\lambda]$ be a monic polynomial, and suppose that all the roots λ of $F(x; \lambda) = 0$ $(x \in B_R)$ has the real parts with the same sign.*

If $g \in W(\theta, T, R)$, then we can define

$$v[F;g](t,x) := \frac{1}{2\pi i} \int_\Gamma \frac{1}{F(x;\zeta)} v[\zeta; g(\cdot, x)](t)\, d\zeta \ ,$$

where $\Gamma = \Gamma_x$ is a simple closed curve in $\boldsymbol{C}$ enclosing all the roots of $F(x;\lambda) = 0$ ($x \in B_R$) and being disjoint with the imaginary axis. (The function $v[F;g]$ is independent of the choice of such Γ.) Further, we have $v[F;g] \in W(\theta, T, R)$ and $F(x;\vartheta)(v[F;g](t,x)) = g(t,x)$.

Proof. For every fixed $x \in B_R$, $g(\cdot, x)$ satisfies the condition (B), and hence $v[\lambda; g(\cdot, x)](t) \in \mathcal{O}(\boldsymbol{C}_\sigma \times B_R \times S_{\theta,T})$, where $\sigma = \pm$ is the sign of the roots of $F(x;\lambda) = 0$. Thus, we can define $v[F;g] \in \mathcal{O}(S_{\theta,T} \times B_R)$. By Lemma 6.6, we have $F(x;\vartheta)(v[F;g](t,x)) = g(t,x)$. By the estimate of $v[\lambda; g]$ in Lemma 6.5, we can easily show that $v[F;g] \in W(\theta, T, R)$. □

Proposition 6.8. *Let $E \in \mathcal{O}(B_R)[\lambda]$ be a monic polynomial. Assume that there exist $L_0 \in \boldsymbol{Z}$ and $\epsilon \geq 0$ such that*

$$\text{if } x \in B_R \text{ and } E(x;\lambda) = 0, \text{ then } L_0 + \epsilon < \operatorname{Re}\lambda < L_0 + \epsilon + 1 \ .$$

For every $L \in \boldsymbol{Z}$, $\theta \in (0, \infty]$, $T' \in (0, T)$, $R' \in (0, R)$, and for every $f \in W^{(L+\epsilon)}(\theta, T', R')$, there exists $u \in W^{(L+\epsilon)}(\theta, T', R')$ such that $E(x;\vartheta)u = f$.

Proof. By putting $f = t^{L+\epsilon} g$ and $u = t^{L+\epsilon} v$, the equation $E(x;\vartheta)u = f$ is equivalent to $E(x;\vartheta + L + \epsilon)v = g$. Since $g \in W(\theta, T', R')$ and since $F(x;\lambda) := E(x;\lambda + L + \epsilon)$ satisfies the assumption of Proposition 6.7, we get $v = v[F;g] \in W(\theta, T', R')$ satisfying $E(x;\vartheta + L + \epsilon)v = g$. □

Last in this section, we consider the operator $\mathcal{C}(x;\vartheta)$, where $\mathcal{C}$ is the indicial polynomial of a Fuchsian partial differential operator P considered in Section 3. We take Γ_l, B_{R_0}, E_l in Lemmata 3.1 and 3.2. Note that $\mathcal{C}(x;\vartheta) = \prod_{l=1}^d E_l(x;\vartheta)$. Put

$$w_{l,p}(t,x) := \frac{1}{2\pi i} \int_{\Gamma_l} \frac{\partial_\lambda^p E_l(x;\lambda)}{E_l(x;\lambda)} t^\lambda \, d\lambda \quad \in \mathcal{O}(S_{\infty,\infty} \times B_{R_0}) \ , \tag{6.5}$$

which correspond to w_p in (6.1) for each E_l.

Let $\theta \in (0, \infty]$ and $T' \in (0, T)$.

Proposition 6.9. (1) *$w_{l,p} \in W^{(L_l+\epsilon)}(\infty, \infty, R_0)$, where $L_l \in \boldsymbol{Z}$ are taken at the beginning of Section 3.*

(2) *For every fixed $x \in B_{R_0}$, the set of functions*

$$\mathcal{B} := \{\, w_{l,p}(\cdot, x) \colon 1 \leq l \leq d,\ l \in \boldsymbol{N};\ 1 \leq p \leq r_l,\ p \in \boldsymbol{N} \,\}$$

is a base of $\operatorname{Ker}_{\mathcal{O}(S_{\theta,T'})} \mathcal{C}(x;\vartheta)$.

(3) *Let $0 < R' < R_0$. If $u \in \widetilde{\mathcal{O}_{\theta,R'}}$ satisfies $\mathcal{C}(x;\vartheta)u = 0$, then there exist*

unique $\varphi_{l,p} \in \mathcal{O}(B_{R'})$ $(1 \le l \le d\ ;\ 1 \le p \le r_l)$ *such that*

$$u(t,x) = \sum_{l=1}^{d} \sum_{p=1}^{r_l} \varphi_{l,p}(x) w_{l,p}(t,x)\ .$$

As a result, $u \in W^{(L_{\min}+\epsilon)}(\infty,\infty,R') \subset \mathcal{O}(S_{\infty,\infty} \times B_{R'})$*, where* $L_{\min} := \min\{ L_l : 1 \le l \le d \}$*. Further, if* $L \in \boldsymbol{Z}$ *and if* $u \in \widetilde{W}^{(L+\epsilon)}(\theta,R')$ *in addition, then* $\varphi_{l,p} = 0$ *for every* l *such that* $L_l < L$*.*

To prove this proposition, we use the following lemma. Recall that we have put $S_l = \mathcal{C}/E_l = \prod_{l' \ne l} E_{l'}$, and that if $l \ne l'$, then the two equations $E_l(x;\lambda) = 0$ and $E_{l'}(x;\lambda) = 0$ have no common root.

Lemma 6.10. *There exist unique* $A_l \in \mathcal{O}(B_{R_0})[\lambda]$ $(1 \le l \le d)$ *such that* $\deg_\lambda A_l \le \deg_\lambda E_l - 1 = r_l - 1$ $(1 \le l \le d)$ *and*

$$\sum_{l=1}^{d} A_l(x;\lambda) S_l(x;\lambda) = 1 \quad \text{for all } x \in B_{R_0} \text{ and all } \lambda \in \boldsymbol{C}\ .$$

Proof. If we fix x, this is well-known. The problem is the holomorphy of the coefficients of A_l in x. The simplest way to show this would be to use the formula

$$A_l(x;\lambda) = \frac{1}{2\pi i} \int_{\Gamma_l} \frac{1}{\mathcal{C}(x;\zeta)} \cdot \frac{E_l(x;\lambda) - E_l(x;\zeta)}{\lambda - \zeta}\, d\zeta\ , \tag{6.6}$$

whose proof is given later (Lemma 6.12). Note that $\dfrac{E_l(x;\lambda) - E_l(x;\zeta)}{\lambda - \zeta}$ is a polynomial of (λ,ζ) with the coefficients in $\mathcal{O}(B_{R_0})$. □

Proof of Proposition 6.9. (1) follows from Proposition 5.3.

(2) We have only to show the independence of $\mathcal{B}$. Suppose that $k_{l,p} \in \boldsymbol{C}$ $(1 \le l \le d;\ 1 \le p \le r_l)$ and $\sum_{l,p} k_{l,p} w_{l,p} = 0$ in $\mathcal{O}(S_{\theta,T})$. We fix $l = l_0$ arbitrarily. By operating $A_{l_0}(x;\vartheta) S_{l_0}(x;\vartheta)$ to this equality, we have $\sum_p k_{l_0,p} A_{l_0}(x;\vartheta) S_{l_0}(x;\vartheta) w_{l_0,p} = 0$, since S_{l_0} is divisible by E_l $(l \ne l_0)$ and since $E_l(x;\vartheta) w_{l,p} = 0$. Since

$$A_{l_0}(x;\lambda) S_{l_0}(x;\lambda) = 1 - F_{l_0}(x;\lambda) E_{l_0}(x;\lambda)$$

for some $F_{l_0} \in \mathcal{O}(B_{R_0})[\lambda]$, we have $A_{l_0}(x;\vartheta) S_{l_0}(x;\vartheta) w_{l_0,p}(t,x) = w_{l_0,p}(t,x)$. Thus, by Proposition 6.3-(1), we have $k_{l_0,p} = 0$ for every p.

(3) By (2), $\varphi_{l,p}(x)$ is uniquely determined for each $x \in B_{R'}$. The problem is the holomorphy in x. Similarly to (1), we have that

$$A_{l_0}(x;\vartheta) S_{l_0}(x;\vartheta) u(t,x) = \sum_{p=1}^{r_{l_0}} \varphi_{l_0,p}(x) w_{l_0,p}(t,x)\ .$$

Hence, by Proposition 6.3-(2), we have $\varphi_{l_0,p} \in \mathcal{O}(B_{R'})$ $(1 \le p \le r_{l_0})$.

Suppose that $L \in \boldsymbol{Z}$ and that $u \in \widetilde{W}^{(L+\epsilon)}(\theta, R')$. We devide the sum representing u into two parts.

$$\begin{aligned} u(t,x) &= \sum_{1\le l\le d; L_l\ge L} \sum_{p=1}^{r_l} \varphi_{l,p}(x) w_{l,p}(t,x) + \sum_{1\le l\le d; L_l+1\le L} \sum_{p=1}^{r_l} \varphi_{l,p}(x) w_{l,p}(t,x) \\ &=: S_1 + S_2 \ . \end{aligned}$$

Since u, $S_1 \in \widetilde{W}^{(L+\epsilon)}(\theta, R')$, we have $S_2 \in \widetilde{W}^{(L+\epsilon)}(\theta, R')$.

Now, Fix $x_0 \in B_{R'}$ and consider $v(t) := S_2(t, x_0)$. Using $v(t) = o(t^{L+\epsilon})$, we shall show $v(t) \equiv 0$, from which the result follows by (2).

Let $E_l(x_0; \lambda) = \prod_{\nu=1}^{h_l} (\lambda - \beta_{l,\nu})^{s_{l,\nu}}$, where $\{\,\beta_{l,\nu}\,\}_\nu$ are distinct. By the well-known fact about partial fractions, we can write

$$\frac{\partial_\lambda^p E_l(x_0;\lambda)}{E_l(x_0;\lambda)} = \sum_{\nu=1}^{h_l} \sum_{\rho=1}^{s_{l,\nu}} \frac{a_{l,p,\nu,\rho}}{(\lambda - \beta_{l,\nu})^\rho} \qquad \text{for some } a_{l,p,\nu,\rho} \in \boldsymbol{C} \ .$$

Hence, we can write $w_{l,p}(t, x_0) = \sum_{\nu=1}^{h_l} \sum_{\rho=1}^{s_{l,\nu}} \frac{1}{(\rho-1)!} a_{l,p,\nu,\rho} t^{\beta_{l,\nu}} (\log t)^{\rho-1}$, and hence $v(t) = \sum_{1\le l\le d; L_l+1\le L} \sum_{\nu=1}^{h_l} \sum_{\rho=1}^{s_{l,\nu}} b_{l,\nu,\rho} t^{\beta_{l,\nu}} (\log t)^{\rho-1}$ for some $b_{l,\nu,\rho} \in \boldsymbol{C}$. Since $\beta_{l,\nu} \in D_l$ by Lemma 3.2-(f), we have $\operatorname{Re} \beta_{l,\nu} < L_l + 1 + \epsilon \le L + \epsilon$, and hence it is now easy to lead $b_{l,\nu,\rho} = 0$ from $v(t) = o(t^{L+\epsilon})$, and hence we have $v(t) \equiv 0$. □

By a repeated use of Proposition 6.8, we have the following.

Proposition 6.11. *For every $L \in \boldsymbol{Z}$, $\theta \in (0, \infty]$, $T' \in (0, T)$, $R' \in (0, R_0)$, and for every $f \in W^{(L+\epsilon)}(\theta, T', R')$, there exists $u \in W^{(L+\epsilon)}(\theta, T', R')$ such that $\mathcal{C}(x; \vartheta) u = f$.*

The author believes that the formula (6.6) is not new, though he found this formula by himself and could not find any references. We give a proof of this formula for the convenience of readers.

Lemma 6.12. *Let $E_1(\lambda)$, ..., $E_d(\lambda) \in \boldsymbol{C}[\lambda]$ be monic polynomials. Suppose that if $l \ne l'$, then $E_l(\lambda) = 0$ and $E_{l'}(\lambda) = 0$ have no common root. Then, for every $g(\lambda) \in \boldsymbol{C}[\lambda]$ with $\deg g \le \sum_{l=1}^d \deg E_l - 1$, there exist unique $A_l(\lambda) \in \boldsymbol{C}[\lambda]$ $(1 \le l \le d)$ such that*

$$g(\lambda) = \sum_{l=1}^d A_l(\lambda) S_l(\lambda) \qquad \text{and} \qquad \deg A_l \le \deg E_l - 1 \quad (1 \le l \le d) \ ,$$

where $S_l(\lambda) := \prod_{1\le l'\le d; l'\ne l} E_{l'}(\lambda)$. Further, if Γ_l is a sum of disjoint simple closed curves in $\boldsymbol{C}$, if all the roots of $E_l(\lambda) = 0$ lie inside Γ_l, and if all the roots of $E_{l'}(\lambda) = 0$ $(l' \ne l)$ lie outside Γ_l, then we have

$$A_l(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_l} \frac{g(\zeta)}{E_1(\zeta) \dots E_d(\zeta)} \cdot \frac{E_l(\zeta) - E_l(\lambda)}{\zeta - \lambda} \, d\zeta \ . \tag{6.7}$$

Note that $\dfrac{E_l(\zeta) - E_l(\lambda)}{\zeta - \lambda}$ is a polynomial of (ζ, λ).

Proof. The first half is well-known, and we have only to show (6.7). Fix $l = l_0$ arbitrarily, and let $E_{l_0}(\lambda) = \prod_{\nu=1}^{p}(\lambda - \alpha_\nu)^{s_\nu}$, where $\{ \alpha_\nu \}_{\nu=1}^{p}$ are distinct. Since

$$\frac{g(\lambda)}{S_{l_0}(\lambda)} = A_{l_0}(\lambda) + E_{l_0}(\lambda) \sum_{1\le l\le d; l\ne l_0} \frac{A_l(\lambda)}{E_l(\lambda)} \ ,$$

we have

$$A_{l_0}^{(j)}(\alpha_\nu) = \frac{d^j}{d\lambda^j}\Big(\frac{g(\lambda)}{S_{l_0}(\lambda)}\Big)\Big|_{\lambda=\alpha_\nu} \quad (1 \le \nu \le p\ ;\ 0 \le j < s_\nu) \ . \tag{6.8}$$

On the other hand, if we put the right hand side of (6.7) as $\widetilde{A}_l(\lambda)$, then this is a polynomial of λ and $\deg \widetilde{A}_l \le \deg E_l - 1$. Further, if λ lies inside Γ_{l_0}, then we have

$$\widetilde{A}_{l_0}(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_{l_0}} \frac{g(\zeta)}{S_{l_0}(\zeta)}\frac{1}{\zeta - \lambda}\, d\zeta - \frac{1}{2\pi i}\int_{\Gamma_{l_0}} \frac{g(\zeta)}{E_{l_0}(\zeta)S_{l_0}(\zeta)}\frac{1}{\zeta-\lambda}\, d\zeta \cdot E_{l_0}(\lambda) \tag{6.9}$$

$$= \frac{g(\lambda)}{S_{l_0}(\lambda)} - h_{l_0}(\lambda)E_{l_0}(\lambda) \ , \tag{6.10}$$

for some $h_{l_0}(\lambda)$ holomophic inside Γ_{l_0}. Hence, we have

$$\widetilde{A}_{l_0}^{(j)}(\alpha_\nu) = \frac{d^j}{d\lambda^j}\Big(\frac{g(\lambda)}{S_{l_0}(\lambda)}\Big)\Big|_{\lambda=\alpha_\nu} \quad (1 \le \nu \le p\ ;\ 0 \le j < s_\nu) \ . \tag{6.11}$$

By (6.8), (6.11), and by $\deg A_{l_0} \le \sum_{\nu=1}^{p} s_\nu - 1$, $\deg \widetilde{A}_{l_0} \le \sum_{\nu=1}^{p} s_\nu - 1$, we have $A_{l_0} = \widetilde{A}_{l_0}$. □

7. Temperedness of solutions

Let P be a Fuchsian partial differential operator considered in Section 3. In this section, we give the *temperedness* of all the solutions of $Pu = 0$, that is, the following proposition.

Proposition 7.1. *Let $\theta \in (0, \infty]$. There exists $a \in \boldsymbol{R}$ such that if $u \in \mathcal{O}(S_{\theta,T} \times B_R)$ satisfies $Pu = 0$, then $u \in W^{(a)}(\theta, T, R)$.*

This follows from a more general result by S. Ōuchi [5]. H. Tahara, however, told the author a simpler proof before Ōuchi's result. We give a sketch of his proof.

Proof. By a change of the variable $t \mapsto t^l$ ($l \in \boldsymbol{N} \setminus \{0\}$), we may assume that P has the following form, without loss of generality.

$$P = \vartheta^m + \sum_{j+|\alpha|\le m; j<m} b_{j,\alpha}(t,x)\vartheta^j (tD_x)^\alpha, \quad b_{j,\alpha} \in \mathcal{O}(\overline{\Delta_T} \times \overline{B_R}) \ . \tag{7.1}$$

By putting $u_{j,\alpha} := \vartheta^j (tD_x)^\alpha u \quad (j + |\alpha| \le m-1)$, the single equation $Pu = 0$ is reduced to a system of equations

$$\vartheta \vec{u} = A(x)\vec{u} + t(B(t,x) + \sum_{j=1}^{n} C_j(t,x) D_{x_j})\vec{u} \ , \tag{7.2}$$

where

$$\vec{u} = (u_{j,\alpha})_{j+|\alpha| \le m-1} \in \boldsymbol{C}^N \ ,$$

and $A(x)$ (resp. $B(t,x)$, $C_j(t,x)$) is an $N \times N$ matrix whose entries belong to $\mathcal{O}(\overline{B_R})$ (resp. $\mathcal{O}(\overline{\Delta_T} \times \overline{B_R})$). Here, N is the number of (j,α) with $j + |\alpha| \le m-1$.

We show that there exists $\alpha > 0$ depending only on $A(x)$ for which if $\vec{u} \in \mathcal{O}(S_{\theta,T} \times B_R; \boldsymbol{C}^N)$ satisfies (7.2), then for every R', R'' with $0 < R'' < R' < R$, there exists $C_0 > 0$ and $T_0 > 0$ such that the following holds.

$$\sup_{|x| \le R''} |\vec{u}(t,x)| \le C_0 \Big(\frac{T_0}{|t|}\Big)^\alpha \sup_{|x| \le R'} |\vec{u}((T_0/|t|)t, x)| \ ,$$
$$\text{for } 0 < |t| \le T_0 \text{ with } |\arg t| < \theta \ .$$

It is clear that this estimate implies the proposition. By rotating t, we may assume that $t \in \boldsymbol{R}$, though we must be careful that C_0 do not depend on $\arg t$.

We show the estimate by solving the Cauchy problem

$$(E) \quad \begin{cases} \vartheta \vec{u} &= A(x)u + t(B(t,x) + \sum_{j=1}^n C_j(t,x) D_{x_j})\vec{u} \ , \\ \vec{u}|_{t=T_0} &= \vec{\varphi}(x) \ , \end{cases}$$

for sufficiently small $T_0 > 0$ by the method of "a scale of Banach spaces". We may assume that $R \le 1$.

We use the following norms.

$$\|\vec{u}\| := \sum_{i=1}^{N} |u_i|, \text{ for } \vec{u} = {}^t(u_1, \dots, u_N) \in \boldsymbol{C}^N \ , \tag{7.3}$$

$$\|A\| := \sup_{\vec{u} \in \boldsymbol{C}^N; \|\vec{u}\| = 1} \|A\vec{u}\|, \text{ for } A \in M_N(\boldsymbol{C}) \ , \tag{7.4}$$

$$\|\vec{u}\|_s := \sup_{|x| \le s} \|\vec{u}(x)\|, \text{ for } \vec{u} \in \mathcal{O}(\overline{B_s}; \boldsymbol{C}^N) \ , \tag{7.5}$$

$$\|A\|_s := \sup_{|x| \le s} \|A(x)\|, \text{ for } A \in \mathcal{O}(\overline{B_s}; M_N(\boldsymbol{C})) \ . \tag{7.6}$$

Put

$$\alpha := \|A\|_R \ ,$$
$$\beta := \sup_{|t| \le T} \|B(t,\cdot)\|_R \ ,$$
$$\gamma := \sum_{j=1}^{n} \sup_{|t| \le T} \|C_j(t,\cdot)\|_R \ ,$$
$$M_0 := \|\vec{\varphi}\|_{R'} \ .$$

Note that $\vec{\varphi}$ is holomorphic in B_R. We solve (E) by the following iterations.

$$(E)_0 \quad \begin{cases} (\vartheta - A(x))\vec{u}_0 = 0 \ , \\ \vec{u}_0|_{t=T_0} = \vec{\varphi}(x) \ , \end{cases}$$

$$(E)_p \quad \begin{cases} (\vartheta - A(x))\vec{u}_p = t(B(t,x) + \sum_{j=1}^n C_j(t,x)D_{x_j})\vec{u}_{p-1} \ , \\ \vec{u}_p|_{t=T_0} = 0 \ , \end{cases} \qquad (p \geq 1) \ .$$

$(E)_0$ can be solved as

$$\vec{u}_0(t,x) = \left(\frac{t}{T_0}\right)^{A(x)} \vec{\varphi}(x) \ , \quad (0 \leq t \leq T_0) \ ,$$

where $z^A := \exp\{(\log z)A\}$. Since we are considering $t \in (0, T_0]$, we have

$$\|\vec{u}_0(t,\cdot)\|_s \leq \left(\frac{T_0}{t}\right)^{\|A\|_s} \|\vec{\varphi}\|_s \leq \left(\frac{T_0}{t}\right)^{\alpha} M_0 \quad (0 < s \leq R') \ .$$

If we put $M := T_0^\alpha M_0$, then we have

$$\|t^\alpha \vec{u}_0(t,\cdot)\|_s \leq M \qquad \text{for } 0 < s \leq R' \ .$$

From this, we have

$$\begin{aligned} \|D_{x_j} t^\alpha \vec{u}_0(t,\cdot)\|_s &\leq \sup_{|x| \leq s} \left| \frac{1}{2\pi i} \int_{\Gamma_{x_j}} \frac{t^\alpha \vec{u}_0(t, x_1, \ldots, \overset{\overset{j}{\vee}}{\zeta}, \ldots, x_n)}{(\zeta - x_j)^2} \, d\zeta \right| \\ &\leq \frac{1}{2\pi} \frac{M}{(R'-s)^2} 2\pi(R'-s) = \frac{M}{R'-s} \quad (0 < s < R') \ , \end{aligned}$$

where Γ_{x_j} is the circle centered at x_j with the radius $R' - s$.

Hence, if we put $\vec{f}_1(t,x) := t(B(t,x) + \sum_{j=1}^n C_j(t,x)D_{x_j})\vec{u}_0$, then we have

$$\begin{aligned} \|t^\alpha \vec{f}_1(t,\cdot)\|_s &\leq t \left\| (B(t,x) + \sum_{j=1}^n C_j(t,x)D_{x_j}) t^\alpha \vec{u}_0(t,\cdot) \right\|_s \\ &\leq t(\beta + \frac{\gamma}{R'-s})M \leq t(\beta+\gamma)\frac{M}{R'-s} \\ &\leq t(\beta+\gamma)\frac{eM}{R'-s} = tK\frac{eM}{R'-s} \qquad \text{for } 0 < s < R' \ , \end{aligned}$$

where $K := \beta + \gamma$.

$(E)_1$ is solved as

$$\vec{u}_1(t,x) = -\int_t^{T_0} \left(\frac{\tau}{t}\right)^{-A(x)} \vec{f}_1(\tau,x) \frac{1}{\tau} \, d\tau \ .$$

Hence, we have

$$\|\vec{u}_1(t,\cdot)\|_s \leq \int_t^{T_0} \left(\frac{\tau}{t}\right)^{\alpha} \|\vec{f}_1(\tau,\cdot)\|_s \frac{1}{\tau} \, d\tau \qquad \text{for } 0 < t \leq T_0 \ .$$

Thus, we have

$$\|t^\alpha \vec{u}_1(t,\cdot)\|_s \leq \int_t^{T_0} \|\tau^\alpha \vec{f}_1(\tau,\cdot)\|_s \frac{1}{\tau} \, d\tau$$

$$\le \int_t^{T_0} \tau K \frac{eM}{R'-s}\frac{1}{\tau}\,d\tau$$
$$\le \frac{MKe}{R'-s}(T_0-t) \quad \text{for } 0<s<R' \ .$$

Now, we show

$$\|t^\alpha \vec{u}_p(t,\cdot)\|_s \le M\Big\{\frac{Ke}{(R'-s)}(T_0-t)\Big\}^p \quad \text{for } 0<s<R' \ , \tag{7.7}$$

by induction on p. If this estimate is valid for p ($p\ge 1$), then we can show that

$$\|D_{x_j} t^\alpha \vec{u}_p(t,\cdot)\|_s \le M\Big\{\frac{Ke}{R'-(s+\frac{R'-s}{p+1})}(T_0-t)\Big\}^p \frac{p+1}{R'-s} \tag{7.8}$$

$$\le M\Big\{\frac{Ke}{R'-s}(T_0-t)\Big\}^p \frac{(p+1)e}{R'-s} \tag{7.9}$$

by a similar argument to the one above, replacing Γ_{x_j} by the circle with the radius $\frac{R'-s}{p+1}$, and using $(1+\frac{1}{p})^p<e$. (This is a standard technique. See, for example, [1], Section 2.2-β.) Hence, we have

$$\|t^\alpha \vec{f}_{p+1}(t,\cdot)\|_s \le tM\frac{(p+1)K^{p+1}e^{p+1}}{(R'-s)^{p+1}}(T_0-t)^p \ ,$$

where $\vec{f}_{p+1}(t,x) := t(B(t,x)+\sum_{j=1}^n C_j(t,x)D_{x_j})\vec{u}_p(t,x)$. From this, we can show that

$$\|t^\alpha \vec{u}_{p+1}(t,\cdot)\|_s \le M\frac{K^{p+1}e^{p+1}}{(R'-s)^{p+1}}(T_0-t)^{p+1} \ ,$$

by estimating

$$\vec{u}_{p+1}(t,x) = -\int_t^{T_0}\Big(\frac{\tau}{t}\Big)^{-A(x)}\vec{f}_{p+1}(\tau,x)\frac{1}{\tau}\,d\tau$$

as above.

Thus, by (7.7), we have

$$\sum_{p=0}^\infty \|\vec{u}_p(t,\cdot)\|_s \le t^{-\alpha}\sum_{p=0}^\infty \|t^\alpha\vec{u}_p(t,\cdot)\|_s \le t^{-\alpha}M\sum_{p=0}^\infty\Big(\frac{KeT_0}{R'-s}\Big)^p$$

for $0<s<R'$ and $0<t\le T_0$. Hence, if we take $0<T_0<\frac{R'-s}{Ke}$, then $\vec{u}(t,x) := \sum_{p=0}^\infty \vec{u}_p(t,x)$ converges on $(0,T_0]\times\overline{B_s}$, and satisfies

$$\|\vec{u}(t,\cdot)\|_s \le \Big(\frac{T_0}{t}\Big)^\alpha \frac{1}{1-\frac{KeT_0}{R'-s}}M_0 \qquad (0<t\le T_0) \ .$$

□

8. Proof of Theorem 3.3-(2)

In this section, we prove Theorem 3.3-(2). First, we prove the injectivity of the solution map (3.4). This also gives the injectivity of (3.8).

We use the following lemma.

Lemma 8.1. *If* $\varphi \in \mathcal{O}(B_{R'})$ $(R' > 0)$, *then we can write*

$$u_{l,p}[\varphi](t,x) = \varphi(x)w_{l,p}(t,x) + t \cdot r_{l,p}[\varphi](t,x) \ ,$$

where $r_{l,p}[\varphi] \in \widetilde{W}^{(L_l+\epsilon)}(\infty, R')$. *Especially,* $u_{l,p}[\varphi] \in \widetilde{W}^{(L_l+\epsilon)}(\infty, R')$.

Proof. We can expand as $\widetilde{V}[\varphi{\cdot}F_{l,p}](t,x;\lambda) = \varphi(x)\frac{\partial_\lambda^p E_l(x:\lambda)}{E_l(x;\lambda)} + t{\cdot}R_{l,p}[\varphi](t,x;\lambda)$, where $R_{l,p}[\varphi] \in \mathcal{O}(\{t=0\} \times B_{R'} \times (\bigcup_{l=1}^{d} \Gamma_l))$. The first term of this expansion produces $\varphi(x)w_{l,p}(t,x)$, Hence, the lemma follows from Propositions 5.3 and 6.9-(1). □

Proof of Injectivity of (3.4). Suppose that $\varphi_{l,p} \in \mathcal{O}(B_{R'})$ $(R' \in (0, R_0)$; $1 \le l \le d$; $1 \le p \le r_l)$ and that $\sum_{l=1}^{d}\sum_{p=1}^{r_l} u_{l,p}[\varphi_{l,p}] = 0$. Further, suppose that there exists (l,p) such that $\varphi_{l,p} \neq 0$, from which we shall lead a contradiction. We can take l_0 as $L_{l_0} = \min\{\, L_l : \varphi_{l,p} \neq 0 \text{ for some } p\,\}$. There exists p such that $\varphi_{l_0,p}(x) \not\equiv 0$.

Take A_l in Lemma 6.10. Recall that $S_{l_0}(x;\vartheta)w_{l,p}(t,x) = 0$ if $l \neq l_0$ and that $A_{l_0}(x;\vartheta)S_{l_0}(x;\vartheta)w_{l_0,p}(t,x) = w_{l_0,p}(t,x)$. Thus, we have

$$0 = \sum_{l=1}^{d}\sum_{p=1}^{r_l} A_{l_0}(x;\vartheta)S_{l_0}(x:\vartheta)u_{l,p}[\varphi_{l,p}](t,x) \tag{8.1}$$

$$= \sum_{p=1}^{r_{l_0}} \varphi_{l_0,p}(x)w_{l_0,p}(t,x) \tag{8.2}$$

$$+ \sum_{l=1}^{d}\sum_{p=1}^{r_l} A_{l_0}(x;\vartheta)S_{l_0}(x:\vartheta)\{t \cdot r_{l,p}[\varphi_{l,p}](t,x)\} \ , \tag{8.3}$$

where $r_{l,p}[\varphi]$ is given in Lemma 8.1. The second sum belongs to $\widetilde{W}^{(L_{l_0}+1+\epsilon)}(\infty, R')$ by Lemma 8.1, and hence we have

$$\sum_{p=1}^{r_{l_0}} \varphi_{l_0,p}(x)w_{l_0,p}(t,x) \in \widetilde{W}^{(L_{l_0}+1+\epsilon)}(\infty, R') \ .$$

By the last part of Lemma 6.9-(3), we have $\varphi_{l_0,p} = 0$ for all p, which contradicts to the definition of l_0. □

Next, we prove the surjectivity of (3.8).

Proof of Surjectivity of (3.8). Suppose that $u \in \widetilde{\mathcal{O}_{\theta,R'}}$ $(\theta \in (0,\infty]$, $R' \in (0, R_0))$ and that $Pu = 0$. Take an arbitrary $R'' \in (0, R')$. Then, there exists $T'' > 0$ such that $u \in \mathcal{O}(S_{\theta,T''} \times B_{R''})$. By Proposition 7.1, there exists $L \in \boldsymbol{Z}$ such that $u \in W^{(L+\epsilon)}(\theta, T'', R'')$. We can write $P = \mathcal{C}(x;\vartheta) + tQ(t,x;\vartheta,D_x)$, where $Q(t,x;\lambda,\xi) \in \mathcal{O}(\overline{\Delta_T}\times\overline{B_R})[\lambda,\xi]$, and we have $\mathcal{C}(x;\vartheta)u = -tQ(t,x;\vartheta,D_x)u \in W^{(L+1+\epsilon)}(\theta,T'',R'')$ by Lemma 5.2.

By Proposition 6.11, there exists $v \in W^{(L+1+\epsilon)}(\theta,T'',R'')$ such that $\mathcal{C}(x;\vartheta)v = \mathcal{C}(x;\vartheta)u$ and hence $u-v \in \mathrm{Ker}_{\mathcal{O}(S_{\theta,T''}\times B_{R''})}\,\mathcal{C}(x;\vartheta)$. Since $u-v \in W^{(L+\epsilon)}(\theta,T'',R'')$, there exists $\varphi_{l,p}[0] \in \mathcal{O}(B_{R''})$ $(1 \le l \le d,\ L_l \ge L\ ;\ 1 \le p \le r_l)$ such that

$$u - v = \sum_{1\le l\le d,\ L_l\ge L;\ 1\le p\le r_l} \varphi_{l,p}[0] w_{l,p}\ ,$$

by Proposition 6.9-(3).

Put $u[1] := u - \sum_{1\le l\le d,\ L_l\ge L;\ 1\le p\le r_l} u_{l,p}[\varphi_{l,p}[0]] \in \widetilde{\mathcal{O}_{\theta,R''}}$. Then, we have $P(u[1]) = 0$. Further, by Lemma 8.1, we have

$$\begin{aligned} u[1] - v &= u[1] - u + u - v \\ &= -\sum_{l;L_l\ge L}\sum_{p=1}^{r_l}\{u_{l,p}[\varphi_{l,p}[0]] - \varphi_{l,p}[0]w_{l,p}\} \in \widetilde{W}^{(L+1+\epsilon)}(\infty,R'')\ . \end{aligned}$$

Hence, we have $u[1] \in \widetilde{W}^{(L+1+\epsilon)}(\theta,R'')$.

Take an arbitrary $R''' \in (0,R'')$. Since $P(u[1]) = 0$, we have $\mathcal{C}(x;\vartheta)u[1] = -tQ(t,x;\vartheta,D_x)u[1] \in \widetilde{W}^{(L+2+\epsilon)}(\theta,R'')$, and hence if we take $R_1 \in (R''',R'')$, then there exists $T_1 > 0$ such that $u[1] \in W^{(L+1+\epsilon)}(\theta,T_1,R_1)$ and $\mathcal{C}(x;\vartheta)u[1] \in W^{(L+2+\epsilon)}(\theta,T_1,R_1)$. By Proposition 6.11, there exists $v[1] \in W^{(L+2+\epsilon)}(\theta,T_1,R_1)$ such that $\mathcal{C}(x;\vartheta)v[1] = \mathcal{C}(x;\vartheta)u[1]$ and hence $u[1]-v[1] \in \mathrm{Ker}_{\mathcal{O}(S_{\theta,T_1}\times B_{R_1})}\,\mathcal{C}(x;\vartheta)$. Since $u[1] - v[1] \in W^{(L+1+\epsilon)}(\theta,T_1,R_1)$, there exists $\varphi_{l,p}[1] \in \mathcal{O}(B_{R_1})$ $(1 \le l \le d,\ L_l \ge L+1\ ;\ 1 \le p \le r_l)$ such that

$$u[1] - v[1] = \sum_{1\le l\le d,\ L_l\ge L+1;\ 1\le p\le r_l} \varphi_{l,p}[1] w_{l,p}\ ,$$

by Proposition 6.9-(3).

Put $u[2] := u[1] - \sum_{1\le l\le d,\ L_l\ge L+1;\ 1\le p\le r_l} u_{l,p}[\varphi_{l,p}[1]] \in \widetilde{\mathcal{O}_{\theta,R_1}}$. Then, we have $P(u[2]) = 0$. Further, in the same way as above by Lemma 8.1, we have $u[2] \in \widetilde{W}^{(L+2+\epsilon)}(\theta,R_1)$.

By repeating this argument, for every $N = 1,2,3,\dots,$ we get $R''' < R_N < R_{N-1} < \cdots < R_1 < R''$ and $u[N] \in \widetilde{W}^{(L+N+\epsilon)}(\theta,R_{N-1})$ that satisfies $P(u[N]) = 0$ and that can be written as $u[N] = u - \sum_{l,p} u_{l,p}[\varphi_{l,p}]$ for some $\varphi_{l,p} \in \mathcal{O}(B_{R_{N-1}})$. By Theorem 4.2, we get $u[N] = 0$ for a sufficiently large N, and hence u can be written as $u = \sum_{l,p} u_{l,p}[\varphi_{l,p}]$ for some $\varphi_{l,p} \in \mathcal{O}(B_{R'''})$.

Since R'' and R''' are arbitrary as long as $0 < R''' < R'' < R'$, and by the injectivity of (3.4), that is, by the uniqueness of $\varphi_{l,p}$, we get $\varphi_{l,p} \in \mathcal{O}(B_{R'})$. $\square$

9. Some variants of the result

We can apply the idea developed above to many problems similar to our problem. In this section, we give two variants of the main theorem.

One is so-called Nagumo-type version. M. S. Baouendi and C. Goulaouic [1] considered not only Fuchsian partial differential operators with holomorphic coefficients, but also those with the coefficients in $C^\infty([0,T];\mathcal{O}(\overline{B_R}))$, and showed theorems corresponding to Theorems 4.1 and 4.2. We can also give a variant of our result for such operators.

Put

$$\widetilde{C^\infty\mathcal{O}} := \operatorname*{indlim}_{T>0,R>0} C^\infty([0,T];\mathcal{O}(\overline{B_R})) \ ,$$

then we can give a linear isomorphism

$$(\mathcal{O}_0)^m \xrightarrow{\sim} \operatorname{Ker}_{\widetilde{C^\infty\mathcal{O}}} P \ ,$$

by the same idea as the main theorem.

In the proof of this version, we must modify the function spaces as

$$\begin{aligned}(9.1)\quad W(\theta,T,R) := \{\, \phi \in C^0([0,T];\mathcal{O}(B_R)) :& \\ \vartheta^l(\phi) \in C^0([0,T];\mathcal{O}(B_R)) \text{ for all } l \in \boldsymbol{N} \,\}& \ ,\end{aligned}$$

and so on. We also need "a priori" regularity similar to Proposition 1 in [1].

The other is the result to Fuchsian *hyperbolic* operators considered by H. Tahara ([6], [7], [8], and so on). In [7], he showed the C^∞ well-posedness of the characteristic Cauchy problems for such operators. In [8], he gave a global linear isomorphism

$$\{C^\infty(\boldsymbol{R}^n)\}^m \xrightarrow{\sim} \operatorname{Ker}_{C^\infty((0,T)\times\boldsymbol{R}^n)} P \ , \tag{9.2}$$

under the assumption that the characteristic exponents do not differ by integer.

Without this assumption, by the same idea as our main theorem, we can give a local linear isomorphism

$$(C_0^\infty)^m \xrightarrow{\sim} \operatorname{Ker}_{\widetilde{C^\infty_{(0,0)}}} P \ ,$$

where $C_0^\infty := \operatorname{indlim}_{R>0} C^\infty(D_R)$ ($D_R := \{\, x \in \boldsymbol{R}^n : |x| < R \,\}$) and $\widetilde{C^\infty_{(0,0)}} := \operatorname{indlim}_{T>0,R>0} C^\infty((0,T)\times D_R)$. Though the argument in the proof of Theorem 3.3-(1) does not work straightly, we can show the existence of $\widetilde{V}[F]$ by using Tahara's result [7, Theorem 3.1] instead of Theorem 4.1, and we can also show the holomorphy of $\widetilde{V}[F]$ in λ by the energy inequality.

Of cource, we also need some modifications to the proof of (2). However, many of Tahara's preliminary results in [8] are valid without any assumptions on the characteristic exponents, which give us necessary tools for the modification.

The author believes that we can give a global isomorphism (9.2) also without any assumption on the characteristic exponents, though he has not succeeded yet.

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