Structure of the solutions to Fuchsian systems

Takeshi MANDAI （萬代武史）

Osaka Electro-Communication University （大阪電気通信大学）

Hidetoshi TAHARA （田原秀敏）

Sophia University （上智大学）

Abstract: To a certain Volevič system of homogeneous singular partial differential equations in a complex domain, called a Fuchsian system, holomorphic solutions which have singularities only on the initial surface are considered. All the solutions are constructed and parametrized in a good way, without any assumptions on the characteristic exponents.

1 Introduction

We consider a system of linear partial differential operators

$$P = tD_tI_m - A(t, x; D_x), \qquad (t, x) \in \boldsymbol{C} \times \boldsymbol{C}^n, \tag{1.1}$$

where I_m is the $m \times m$ unit matrix, and

$$A = \big(A_{i,j}(t, x; D_x)\big)_{1\le i,j\le m}\ , \qquad A_{i,j} = \sum_{\alpha:\text{finite}} a_{i,j;\alpha}(t, x)D_x^\alpha\ ,$$

$a_{i,j;\alpha}$ are holomorphic in a neighborhood of the origin $(0, 0) \in \boldsymbol{C}^{n+1}$. We use $D_t = \dfrac{\partial}{\partial t}$, $D_x = (D_1, \dots, D_n)$, $D_j = \dfrac{\partial}{\partial x_j}$, without dividing by $\sqrt{-1}$.

P is called a *Fuchsian system* if P satisfies the following two conditions.

The research was partially supported by the Ministry of Education, Science, Sports and Culture (Japan), Grant-in-Aid for Scientific Research 12640194(2000,2001).

Key Words: Fuchsian system, Volevič system, regular singularity, characteristic exponent, characteristic index.

(A-1) There exists $n_j \in \boldsymbol{N} := \{0, 1, 2, \dots\}$ such that

$$\mathrm{ord}_{D_x} A_{i,j}(t, x; D_x) \le n_i - n_j + 1 \ .$$

(A-2) $A(0, x; D_x) =: A_0(x)$ is independent of D_x.

The condition (A-1) is equivalent to each of the following condition ([4], [5]).

(A-1)' $$\max_{\substack{1\le p\le m \\ 1\le i_1<\cdots<i_p\le m}} \left(\frac{1}{p} \max_{\pi\in\mathcal{S}_p} \sum_{k=1}^{p} \mathrm{ord}_{D_x} A_{i_k, i_{\pi(k)}} \right) =: \rho(A) \le 1$$

($\rho(A)$ is called the *matrix order* of A.)

When this condition is satisfied, the system $D_t I_m - A(t, x; D_x)$ is called a *kowalevskian system in Volevič's sense* ([4]).

The polynomial

$$\mathcal{C}(x; \lambda) := \det\bigl(\lambda I_m - A_0(x)\bigr)$$

of λ is called the *indicial polynomial* of P, and a root λ of $\mathcal{C}(x; \lambda) = 0$ is called a *characteristic exponent* or a *characteristic index* of P at x.

The second author ([6], [7]) has shown the following fundamental theorems corresponding to the Cauchy-Kowalevsky theorem and the Holmgren theorem. Let $\mathcal{O}_{(0,0)}$ denote the germ space of holomorphic functions at $(0, 0) \in \boldsymbol{C} \times \boldsymbol{C}^n$.

Theorem 1.1 ([6, Theorem 1.2.10]). *If $\mathcal{C}(0; j) \ne 0$ ($j \in \boldsymbol{N}$), then for every $\overrightarrow{f} \in (\mathcal{O}_{(0,0)})^m$, there exists a unique $\overrightarrow{u} \in (\mathcal{O}_{(0,0)})^m$ such that $P\overrightarrow{u} = \overrightarrow{f}(t, x)$.*

Theorem 1.2 ([7, Theorem 2]). *Let Ω be an open neighborhood of $0 \in \boldsymbol{R}^n$ and $T > 0$. Let $L \in \boldsymbol{R}$ satisfy that if $\mathcal{C}(x; \lambda) = 0$ ($x \in \Omega$), then $\mathrm{Re}\,\lambda < L$. If $\overrightarrow{u}(t) = \overrightarrow{u}(t, x) \in C^1\bigl((0, T], \mathcal{D}'(\Omega)\bigr)^m$ satisfies $P\overrightarrow{u} = \overrightarrow{0}$ in $(0, T) \times \Omega$, and if $t^{-L}\overrightarrow{u} \in C^0\bigl([0, T], \mathcal{D}'(\Omega)\bigr)^m$, then $\overrightarrow{u} = \overrightarrow{0}$ near $(0, 0)$ in $(0, T) \times \Omega$. Here, $\mathcal{D}'(\Omega)$ denotes the space of Schwartz distributions on Ω.*

Now, we introduce the following notation.

$$\mathcal{O}(\Omega) : = \{\, \text{holomorphic functions on } \Omega \,\} \ ,$$

$$B_R := \{\, x \in \boldsymbol{C}^n : |x| < R \,\} \ , \qquad \Delta_T := \{\, t \in \boldsymbol{C} : |t| < T \,\} \quad (T > 0) \ ,$$
$$\mathcal{O}_0 := \bigcup_{R>0} \mathcal{O}(B_R) \ , \qquad \mathcal{O}_{(0,0)} := \bigcup_{R>0, T>0} \mathcal{O}(\Delta_T \times B_R) \ ,$$
$$S_{\infty,T} := \mathcal{R}(\Delta_T \setminus \{0\}) \quad \text{(the universal covering of } \Delta_T \setminus \{0\}) \ ,$$
$$S_{\theta,T} := \{\, t \in S_{\infty,T} : |\arg t| \le \theta \,\} \ , \qquad \widetilde{\mathcal{O}} := \bigcup_{T>0, R>0} \mathcal{O}(S_{\infty,T} \times B_R) \ .$$

Now, we consider solutions of $P\overrightarrow{u} = \overrightarrow{0}$ which are singular only at $t = 0$, that is, $\overrightarrow{u} \in (\widetilde{\mathcal{O}})^m$. Under the assumption that the characteristic exponents $\lambda_j(x)$ $(j = 1, 2, \ldots, m)$ of P do *not* differ by integers, that is, $\lambda_i(0) - \lambda_j(0) \notin \boldsymbol{Z}$ $(i \neq j)$, the structure of the kernel $\operatorname{Ker}_{(\widetilde{\mathcal{O}})^m} P$ of the map $P : (\widetilde{\mathcal{O}})^m \to (\widetilde{\mathcal{O}})^m$ has been studied by the second author([6]).

Our purpose of this talk is to construct a solution map, that is, a linear isomorphism

$$(\mathcal{O}_0)^m \xrightarrow{\sim} \operatorname{Ker}_{(\widetilde{\mathcal{O}})^m} P := \{\, \overrightarrow{u} \in (\widetilde{\mathcal{O}})^m : P\overrightarrow{u} = \overrightarrow{0} \,\} \ , \tag{1.2}$$

rather explicitly, with *no* assumptions on the characteristic exponents (Theorem 2.2).

In the case of *single* Fuchsian partial differential equations, the first author([2]) have constructed a good solution map. These single equations can be reduced to our Fuchsian systems as follows.

Remark 1.3. Let P' be a single Fuchsian partial differential operator with weight 0 ([1], [6], [2], etc.); that is, $P' = (tD_t)^m + \sum_{j=1}^m P'_j(t, x; D_x)(tD_t)^{m-j}$, $\operatorname{ord}_{D_x} P'_j \le j$, and $P'_j(0, x; D_x) =: a_j(x)$ is a function of x. Then, by $u_j = (tD_t)^{j-1} u$ $(1 \le j \le m)$, the equation $Pu = f$ is reduced to

$$\left(tD_t I_m - \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -P'_m & -P'_{m-1} & -P'_{m-2} & \ldots & -P'_1 \end{pmatrix} \right) \overrightarrow{u} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{pmatrix} \ .$$

Since this system satisfies (A-1) with $n_j = j$ and (A-2), it is a Fuchsian system. Further, this system has the same indicial polynomial $\mathcal{C}(x; \lambda)$ as P', where the indicial polynomial of P' is defined by

$$\mathcal{C}[P'](x;\lambda) := \lambda^m + \sum_{j=1}^{m} a_j(x)\lambda^{m-j} = \left[t^{-\lambda}P'(t^\lambda)\right]|_{t=0} \ .$$

2 Construction of the solution map

Let μ_l ($l = 1, \dots, d$) be all the distinct roots of $\mathcal{C}(0;\lambda) = 0$, and let r_l be the multiplicity of μ_l. There exists $Q(x) \in GL_m(\mathcal{O}_0)$ such that

- $Q(x)^{-1}A_0(x)Q(x) = A_1(x) \otimes \cdots \otimes A_d(x) := \begin{pmatrix} A_1(x) & O & \dots & O \\ O & A_2(x) & O & \vdots \\ \vdots & O & \ddots & \vdots \\ O & \dots & O & A_d(x) \end{pmatrix}$,
- $A_l \in M_{r_l}(\mathcal{O}_0)$ ($l = 1, \dots, d$),
- $\det\bigl(\lambda I_{r_l} - A_l(0)\bigr) = (\lambda - \mu_l)^{r_l}$ ($l = 1, \dots, d$).

Corresponding to the blocks of $Q(x)^{-1}A_0(x)Q(x)$, we denote the l-th block of $\overrightarrow{u}$ by $\overrightarrow{u}^{\flat(l)} \in \boldsymbol{C}^{r_l}$, that is, $\overrightarrow{u} = \begin{pmatrix} \overrightarrow{u}^{\flat(1)} \\ \vdots \\ \overrightarrow{u}^{\flat(d)} \end{pmatrix}$. Conversely, for an r_l-vector $\overrightarrow{v} \in \boldsymbol{C}^{r_l}$, we denote by $\overrightarrow{v}^{\sharp(l)} \in \boldsymbol{C}^m$ the m-vector

$$\overrightarrow{v}^{\sharp(l)} = \begin{pmatrix} 0 \\ \vdots \\ \overrightarrow{v} \\ \vdots \\ 0 \end{pmatrix} \langle l \text{ th block}$$

with the entries $\overrightarrow{v}$ in the l-th block and the entries 0 in the other blocks.

Set

$$\Lambda_P := \{\, \mu_l - j \in \boldsymbol{C} : 1 \le l \le d, j \in \boldsymbol{N} \,\} \ . \tag{2.1}$$

Take $\epsilon \ge 0$ as $\operatorname{Re}\mu_l - \epsilon \notin \boldsymbol{Z}$ for all l. For each l, take $L_l \in \boldsymbol{Z}$ as $L_l + \epsilon < \operatorname{Re}\mu_l < L_l + \epsilon + 1$.

Lemma 2.1. (1) *For each l, there exists a domain D_l in $\boldsymbol{C}$ enclosed by a simple closed curve Γ_l such that*

(a) $\mu_l \in D_l \ (1 \le l \le d)$,

(b) $\overline{D_l} \cap \overline{D_{l'}} = \emptyset \ (l \neq l')$, *where $\overline{D}$ denotes the closure of D.*

(c) $\overline{D_l} \cap \Lambda_P = \{ \mu_l \}$ *for every l.*

(d) $\overline{D_l} \subset \{ \lambda \in \boldsymbol{C} : L_l + \epsilon < \operatorname{Re} \lambda < L_l + \epsilon + 1 \}$ *for every l.*

(2) *There exists $R_0 > 0$ such that*

(e) $\mathcal{C}(x; \lambda + j) \neq 0$ *for every $x \in B_{R_0}$, every $\lambda \in \bigcup_{l=1}^{d} \Gamma_l$, and every $j \in \boldsymbol{N}$.*

The main result is

Theorem 2.2. *For every l and every $\overrightarrow{\varphi_l} \in (\mathcal{O}_0)^{r_l}$, there exists a unique $\overrightarrow{V} = \overrightarrow{V}[l, \overrightarrow{\varphi_l}](t, x; \lambda) \in \mathcal{O}(\{(0,0)\} \times (\bigcup_{l=1}^{d} \Gamma_l))^m$ such that*

$$P(t^\lambda \overrightarrow{V}) = t^\lambda Q(x) \overrightarrow{\varphi_l}^{\sharp(l)}(x) \tag{2.2}$$

in a neighborhood of $\{(0,0)\} \times \left(\bigcup_{l=1}^{d} \Gamma_l \right)$.

Set $\overrightarrow{u_l}[\overrightarrow{\varphi_l}](t, x) := \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{\Gamma_l} t^\lambda \overrightarrow{V}[l, \overrightarrow{\varphi_l}](t, x; \lambda)\, d\lambda$. Then, the map

$$(\mathcal{O}_0)^m \ni \begin{pmatrix} \overrightarrow{\varphi_1} \\ \vdots \\ \overrightarrow{\varphi_d} \end{pmatrix} \overset{\sim}{\longmapsto} \sum_{l=1}^{d} \overrightarrow{u_l}[\overrightarrow{\varphi_l}] \in \operatorname{Ker}_{(\tilde{\mathcal{O}})^m} P \tag{2.3}$$

is a linear isomorphism.

3 Expansion of the solutions

Expand the operator A and the vector $\overrightarrow{V}$ as follows.

$$A(t, x; D_x) = A_0(x) + \sum_{l=1}^{\infty} t^l B_l(x; D_x) \ ,$$

$$\overrightarrow{V}[l,\overrightarrow{\varphi_l}](t,x;\lambda)=\sum_{j=0}^{\infty}t^j\overrightarrow{V_j}(x;\lambda) \ .$$

Then, the equation (2.2) for $\overrightarrow{V}$ is equivalent to

$$\big(\lambda I_m - A_0(x)\big)\overrightarrow{V_0}(x;\lambda)=Q(x)\overrightarrow{\varphi_l}(x) \ , \tag{3.1}$$

$$\big((\lambda+j)I_m - A_0(x)\big)\overrightarrow{V_j}(x;\lambda)=\sum_{l=1}^{j}B_l(x;D_x)\overrightarrow{V_{j-l}}(x;\lambda) \quad (j\geq 1) \ . \tag{3.2}$$

From these equations, we can determine $\overrightarrow{V_j}$ by Lemma 2.1 (e), and we get an expansion of $\overrightarrow{u_l}[\overrightarrow{\varphi_l}]$ as follows.

$$\overrightarrow{u_l}[\overrightarrow{\varphi_l}](t,x)=\sum_{j=0}^{\infty}t^j\overrightarrow{u_{l,j}}(t,x) \ ,$$

$$\overrightarrow{u_{l,j}}(t,x):=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma}t^{\lambda}\overrightarrow{V_j}(x;\lambda)\,d\lambda \ .$$

Especially, the leading term of $\overrightarrow{u_l}[\overrightarrow{\varphi_l}]$ is

$$\overrightarrow{u_{l,0}}(t,x)=t^{A_0(x)}Q(x)\overrightarrow{\varphi_l}^{\sharp(l)}(x)=Q(x)\{t^{A_l(x)}\overrightarrow{\varphi_l}(x)\}^{\sharp(l)} \ . \tag{3.3}$$

4 Sketch of the proof of the existence of $\overrightarrow{V}$

We change the letter λ to ζ. Then, the system $P(t^{\zeta}\overrightarrow{V})=t^{\zeta}Q(x)\overrightarrow{\varphi_l}^{\sharp(l)}(x)$ is equivalent to another system

$$\widetilde{P}\overrightarrow{V}:=\big((tD_t+\zeta)I_m - A(t,x;D_x)\big)\overrightarrow{V}=Q(x)\overrightarrow{\varphi_l}^{\sharp(l)}(x) \ .$$

This is also a Fuchsian system in (t,x,ζ). Note that we consider (x,ζ) as the space variables. Further, the indicial polynomial of $\widetilde{P}$ is

$$\mathcal{C}[\widetilde{P}](x,\zeta;\lambda)=\mathcal{C}[P](x;\lambda+\zeta) \ .$$

Since $\mathcal{C}[\widetilde{P}](0,\zeta;j)\neq 0$ $(\zeta\in\Gamma_l, \ j\in\boldsymbol{N})$ by Lemma 2.1 (e), we can use Theorem 1.1 to this new system. Thus, there exists a unique $\overrightarrow{V}=\overrightarrow{V}[l,\overrightarrow{\varphi_l}](t,x;\zeta)\in\mathcal{O}(\{(0,0)\}\times\Gamma_l)^m$ such that $\widetilde{P}\overrightarrow{V}=Q(x)\overrightarrow{\varphi_l}^{\sharp(l)}(x)$.

5 Function spaces to estimate the order

Definition 5.1. ([2, Definition 5.1]) For $a \in \boldsymbol{R}$, we set

$$W^{(a)} := \bigcup_{R>0,\ T>0} \Big\{ \phi \in \mathcal{O}(S_{\infty,T} \times B_R) : \\ \sup_{|x|<R} |\phi(t,x)| \to 0 \ \ (\text{as } t \to 0 \text{ in } S_{\theta,T}) \text{ for every } \theta > 0 \Big\}$$

Lemma 5.2. ([2, Lemma 5.2]) (1) $a' < a \implies W^{(a)} \subset W^{(a')}$.

(2) $t \times W^{(a)} \subset W^{(a+1)}$, $\quad \partial_t(W^{(a)}) \subset W^{(a-1)}$.

(3) *If $B(t,x;D_x)$ is a partial differential operator in x with $\mathcal{O}_{(0,0)}$ coefficients, then $B(t,x;D_x)(W^{(a)}) \subset W^{(a)}$.*

6 Keys to the proof of the theorem

The **first key** is the temperedness of the solutions in $(\widetilde{\mathcal{O}})^m$.

Proposition 6.1. *There exists $a \in \boldsymbol{R}$ such that if $\overrightarrow{u} \in (\widetilde{\mathcal{O}})^m$ and $P\overrightarrow{u} = \overrightarrow{0}$, then $\overrightarrow{u} \in (W^{(a)})^m$.*

The **second key** is an estimate of the remainder terms of our solutions $\overrightarrow{u_l}[\overrightarrow{\varphi_l}](t,x)$.

Lemma 6.2. *For $\overrightarrow{\varphi_l} \in (\mathcal{O}_0)^{r_l}$, we have*

$$\overrightarrow{u_l}[\overrightarrow{\varphi_l}](t,x) = Q(x)\{t^{A_l(x)}\overrightarrow{\varphi_l}(x)\}^{\sharp(l)} + t\cdot \overrightarrow{r_l}[\overrightarrow{\varphi_l}](t,x) \ \ ,$$

and $\overrightarrow{r_l}[\overrightarrow{\varphi_l}] \in (W^{(L_l+\epsilon)})^m$. Note that $t^{A_l(x)}\overrightarrow{\varphi_l}(x) \in (W^{(L_l+\epsilon)})^{r_l}$ and $\overrightarrow{u_l}[\overrightarrow{\varphi_l}] \in (W^{(L_l+\epsilon)})^m$.

The **third key** is the two facts on the Euler system $\big(tD_t - A_0(x)\big)\overrightarrow{u} = \overrightarrow{f}(t,x)$ with holomorphic parameters x.

Lemma 6.3. *If $\overrightarrow{u} \in (\widetilde{\mathcal{O}})^m$ and $\big(tD_tI_m - A_0(x)\big)\overrightarrow{u} = \overrightarrow{0}$, then there exists $\overrightarrow{\varphi_l} \in (\mathcal{O}_0)^{r_l}$ $(1 \le l \le d)$ such that*

$$\overrightarrow{u} = \sum_{l=1}^{d} Q(x)\{t^{A_l(x)}\overrightarrow{\varphi_l}(x)\}^{\sharp(l)} = Q(x)\begin{pmatrix} t^{A_1(x)}\overrightarrow{\varphi_1}(x) \\ \vdots \\ t^{A_d(x)}\overrightarrow{\varphi_d}(x) \end{pmatrix} \ .$$

Further, if $L \in \boldsymbol{Z}$ and $\overrightarrow{u} \in \widetilde{W}^{(L+\epsilon)}(\theta, R)^m$, then $\overrightarrow{\varphi_l} = 0$ for all l such that $L_l < L$.

Proposition 6.4. *For any $L \in \boldsymbol{Z}$ and any $\overrightarrow{g} \in W^{(L+\epsilon)}$, there exists $\overrightarrow{v} \in W^{(L+\epsilon)}$ such that $\big(tD_t I_m - A_0(x)\big)\overrightarrow{v} = \overrightarrow{g}(t, x)$.*

If a root $\lambda(x)$ of $\mathcal{C}(x;\lambda) = 0$ touches the line $\operatorname{Re}\lambda = L + \epsilon$ in λ-plane, then this proposition does not hold, as the simplest example $tD_t v = \dfrac{1}{\log t}$ shows ($m = 1$, $L+\epsilon = 0$, no parameter x). This proposition is the reason why we took ϵ.

7 Proof of the injectivity of the solution map

Assume that $\overrightarrow{\varphi_l} \in (\mathcal{O}_0)^{r_l}$ $(1 \le l \le d)$, $\displaystyle\sum_{l=1}^{d} \overrightarrow{u_l}[\overrightarrow{\varphi_l}] = \overrightarrow{0}$, and that there exists l such that $\overrightarrow{\varphi_l} \neq \overrightarrow{0}$. Take l_0 as $L_{l_0} = \min\{\, L_l : \overrightarrow{\varphi_l} \neq \overrightarrow{0}\,\}$.

For each l with $\overrightarrow{\varphi_l} \neq \overrightarrow{0}$, consider $(\overrightarrow{v_l})^{\flat(l_0)}$: the l_0-th block of $\overrightarrow{v_l} := Q^{-1}\overrightarrow{u_l}[\overrightarrow{\varphi_l}]$. Then, we have by Lemma 6.2

$$(\overrightarrow{v_{l_0}})^{\flat(l_0)} = t^{A_{l_0}(x)}\overrightarrow{\varphi_{l_0}}(x) + (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}} \quad .$$

On the other hande, if $l \neq l_0$, then $L_l \ge L_{l_0}$ and hence

$$(\overrightarrow{v_l})^{\flat(l_0)} \in (W^{(L_l+1+\epsilon)})^{r_{l_0}} \subset (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}} \quad .$$

Thus,

$$\overrightarrow{0} = \sum_{l=1}^{d}(Q^{-1}\overrightarrow{u_l}[\overrightarrow{\varphi_l}])^{\flat(l_0)} = t^{A_{l_0}(x)}\overrightarrow{\varphi_{l_0}}(x) + (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}} \quad .$$

Namely, $t^{A_{l_0}(x)}\overrightarrow{\varphi_{l_0}}(x) \in (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}}$. It is easy to show that this implies $\overrightarrow{\varphi_{l_0}} = \overrightarrow{0}$, which contradicts the definition of l_0.

8 Proof of the surjectivity of the solution map

Let $\overrightarrow{u} \in (\widetilde{\mathcal{O}})^m$ and $P\overrightarrow{u} = \overrightarrow{0}$. Decompose $A(t, x; D_x) = A_0(x) + tB(t, x; D_x)$.

(I) By Proposition 6.1, there exists $L \in \boldsymbol{Z}$ such that $\underline{\overrightarrow{u} \in (W^{(L+\epsilon)})^m}$.

By Lemma 5.2, we have $tB(\overrightarrow{u}) \in (W^{(L+1+\epsilon)})^m$.

(II) By Proposition 6.4, there exists $\overrightarrow{v} \in (W^{(L+1+\epsilon)})^m$ such that $\big(tD_t I_m - A_0(x)\big)\overrightarrow{v} = tB(\overrightarrow{u}) = \big(tD_t I_m - A_0(x)\big)\overrightarrow{u}$.

(III) Since $\big(tD_t I_m - A_0(x)\big)(\overrightarrow{u} - \overrightarrow{v}) = \overrightarrow{0}$ and $\overrightarrow{u} - \overrightarrow{v} \in (W^{(L+\epsilon)})^m$, there exists $\overrightarrow{\varphi_l}[1] \in (\mathcal{O}_0)^{r_l}$ such that $\overrightarrow{\varphi_l}[1] = \overrightarrow{0}$ if $L_l < L$, and that

$$\overrightarrow{u} - \overrightarrow{v} = \sum_{l=1}^{d} Q(x)\{t^{A_l(x)}\overrightarrow{\varphi_l}[1](x)\}^{\sharp(l)} \quad .$$

by Lemma 6.3.

(IV) Set

$$\overrightarrow{u}[1] := \overrightarrow{u} - \sum_{l=1}^{d} \overrightarrow{u_l}\left[\overrightarrow{\varphi_l}[1]\right] \in (W^{(L+1+\epsilon)})^m \quad .$$

Then, we have $P(\overrightarrow{u}[1]) = \overrightarrow{0}$, $\overrightarrow{u}[1] \in (W^{(L+1+\epsilon)})^m$.

Now, we can return to the step (I) taking $\overrightarrow{u}[1]$ instead of $\overrightarrow{u}$, with order $L+1+\epsilon$ instead of $L+\epsilon$.

Repeating such arguments, we have $\overrightarrow{\varphi_l}[j] \in (\mathcal{O}_0)^{r_l}$ $(j = 2, 3, \dots)$ such that $\overrightarrow{\varphi_l}[j] = \overrightarrow{0}$ if $L_l < L + j - 1$, and that

$$\overrightarrow{u}[j] := \overrightarrow{u}[j-1] - \sum_{l=1}^{d} \overrightarrow{u_l}\left[\overrightarrow{\varphi_l}[j]\right] \left(= \overrightarrow{u} - \sum_{k=1}^{j}\sum_{l=1}^{d} \overrightarrow{u_l}\left[\overrightarrow{\varphi_l}[k]\right]\right) \in (W^{(L+j+\epsilon)})^m \quad ,$$

and $P(\overrightarrow{u}[j]) = \overrightarrow{0}$.

By Theorem 1.2, we have $\overrightarrow{u}[M] = \overrightarrow{0}$ for sufficiently large M. Thus, we get $\overrightarrow{\varphi_l} := \sum_{k=1}^{M} \overrightarrow{\varphi_l}[k] \in (\mathcal{O}_0)^{r_l}$.

References

[1] M. S. Baouendi and C. Goulaouic, Cauchy problems with characteristic initial hypersurface, *Comm. Pure Appl. Math.* **26** (1973), 455–475.

[2] T. Mandai, The method of Frobenius to Fuchsian partial differential equations, *J. Math. Soc. Japan* **52** (2000), 645–672.

[3] T. Mandai and H. Tahara, Structure of solutions to Fuchsian systems of partial differential equations, to appear in *Nagoya Math. J.*.

[4] M. Miyake, On Cauchy-Kowalevski's theorem for general systems, *Publ. RIMS, Kyoto Univ.*, **15**(1979), 315–337.

[5] S. Mizohata, On Kowalewskian systems, *Russ. Math. Surveys*, **29** (1974), 223–235.

[6] H. Tahara, Fuchsian type equations and Fuchsian hyperbolic equations, *Japan. J. Math.* (*N.S.*) **5** (1979), 245–347.

[7] ———, On a Volevič system of singular partial differential equations, *J. Math. Soc. Japan* **34** (1982), 279–288.