Structure of Distribution Null-Solutions to Fuchsian Partial Differential Equations

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Abstract. We give a structure theorem for distribution null-solutions to Fuchsian partial differential equations in the sense of M. S. Baouendi and C. Goulaouic. We assume neither that the characteristic exponents are real-analytic nor that the charactersitic exponents do not differ by integer.

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1 Introduction

Consider a *Fuchsian partial differential operator* with weight $\omega := m - k$ in the sense of M. S. Baouendi and C. Goulaouic([1]) :

$$P = t^k \partial_t^m + \sum_{j=1}^{k} a_j(x) t^{k-j} \partial_t^{m-j} + \sum_{l<m} \sum_{|\alpha| \le m-l} b_{l,\alpha}(t,x) t^{d(l)} \partial_t^l \partial_x^\alpha \ , \qquad (1)$$

$$0 \le k \le m, \quad d(l) := \max\{\, 0, l-m+k+1 \,\}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n \ .$$

When $m = k$ ($\omega = 0$), M. Kashiwara and T. Oshima ([5], Definition 4.2) called such an operator "*an operator which has regular singularity in a weak sense along* $\Sigma_0 := \{\, t = 0 \,\}$."

In the following two categories (coefficients, data, solutions) :

(a) functions real-analytic in (t, x),
(b) functions real-analytic in x and of class C^∞ in t,

M. S. Baouendi and C. Goulaouic showed the following results:

A. the unique solvability of the characteristic Cauchy problems (Cauchy-Kovalevsky type theorem, Nagumo type theorem),

B. the uniqueness in a wider class of solutions (Holmgren type theorem).

H. Tahara([8], and so on) also showed similar results to A and B in the category of

(c) functions of class C^∞ in (t, x),

for "*Fuchsian hyperbolic operators*", which are, roughly speaking, operators being weakly hyperbolic in $t > 0$ and satisfying "Levi conditions".

In all cases, it easily follows that there exist *no* sufficiently smooth null-solutions. Here, a distribution u near $(t, x) = (0, 0)$ is called a *null-solution* for P, if $Pu = 0$ near $(0, 0)$ and if $(0, 0) \in \operatorname{supp} u \subset \Sigma_+ := \{ t \geq 0 \}$, where $\operatorname{supp} u$ denotes the support of u.

K. Igari([4]) showed the existence of a distribution null-solution under a weak additional condition in Case (a). This solution is real-analytic in x. The author([6]) showed the existence of a distribution null-solution under *no* additional conditions in Case (a),(b),(c). This solution is also real-analytic (Case (a),(b)) or of class C^∞ (Case (c)) in x.

The aim of this study is to make the structure of all solutions belonging to these classes as clear as possible. We consider the case (b) for simplicity. Namely, the coefficients of P are of class C^∞ in t and real-analytic in x.

Many problems about Fuchsian partial differential equations have been considered by many authors. Almost all of them, however, have some assumptions on the characteristic exponents(indices), especially the one that the characteristic exponents do not differ by integer. In this study, we assume neither that the characteristic exponents(indices) are real-analytic in x nor that the characteristic exponents do not differ by integer.

NOTATION:

(i) The set of all integers (resp. nonnegative integers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{N}$).
(ii) The real part of a complex number z is denoted by $\mathrm{Re} z$.
(iii) Put $\vartheta := t\partial_t$ and $(\lambda)_l := \prod_{j=0}^{l-1}(\lambda - j)$ for $l \in \mathbb{N}$.
(iv) For a domain Ω in $\mathbb{C}^n$, we denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions on Ω. For a complete locally convex topological vector space X, we put $\mathcal{O}(\Omega; X) := \{ f \in C^0(\Omega; X) \mid \langle \phi, f \rangle_X \in \mathcal{O}(\Omega) \text{ for every } \phi \in X' \}$, where X' is the dual space of X and $\langle \cdot, \cdot \rangle_X$ denotes the duality between X' and X. Note that if Ω is a domain in $\mathbb{C}^l$ and D is a domain in $\mathbb{R}^n$, then $\mathcal{O}(\Omega; C^\infty(D)) = C^\infty(D; \mathcal{O}(\Omega))$.
(v) The space of test functions on an open interval I of $\mathbb{R}$ is denoted by $\mathcal{D}(I)$ and the space of distributions by $\mathcal{D}'(I)$. The space of rapidly decreasing C^∞ functions is denoted by $\mathcal{S}(\mathbb{R})$ and the space of tempered distributions by $\mathcal{S}'(\mathbb{R})$. The duality between each pair of these spaces is denoted by $\langle \cdot, \cdot \rangle$. More generally, for a complete locally convex topological vector space X, the space of all X-valued distributions is denoted by $\mathcal{D}'(I; X)$, which is defines as $\mathcal{L}(\mathcal{D}(I), X)$, where $\mathcal{L}(X, Y)$ denotes the space of all continuous linear mappings from X to Y (See [7]). Note that $\mathcal{D}'(I; \mathcal{O}(\Omega)) = \mathcal{O}(\Omega; \mathcal{D}'(I))$.

Put $\mathcal{D}'_+(I;X) := \{\, f \in \mathcal{D}'(I;X) \mid f(t) = 0 \text{ in } X \text{ for } t < 0 \,\}$. Also, for $N \in \mathbb{N}$ put

$$C_+^N(I;X) : = \{\, f \in C^N(I;X) \mid f(t) = 0 \text{ in } X \text{ for } t < 0 \,\},$$
$$C_+^{-N}(I;X) : = \{\, \partial_t^N(f) \in \mathcal{D}'_+(I;X) \mid f \in C_+^0(I;X) \,\} \ .$$

(vi) For $z \in \mathbb{C}$ with $\mathrm{Re}z > -1$, we put

$$t_+^z := \begin{cases} t^z \ (t > 0) \\ 0 \ \ (t \le 0) \end{cases} ,$$

which is a locally integrable function of t with holomorphic parameter z, and hence belongs to $\mathcal{D}'_+(\mathbb{R}; \mathcal{O}(\{\, z \in \mathbb{C} \mid \mathrm{Re}z > -1 \,\}))$. By $\partial_t(t_+^z) = z t_+^{z-1}$, this distribution t_+^z is extended to $z \in \mathbb{C} \setminus \{\, -1, -2, \dots \}$ meromorphically with simple poles at $z = -1, -2, \dots$ ([3]).

(vii) For a commutative ring R, the ring of polynomials of λ with the coefficients belonging to R is denoted by $R[\lambda]$. The degree of $F(\lambda) \in R[\lambda]$ is denoted by $\deg_\lambda F$.

2 Review of some Results for Ordinary Differential Equations

In this section, we review some results for ordinary differential equations with a regular singularity at $t = 0$, which will help us to understand our result.

Consider

$$P = t^k \partial_t^m + \sum_{j=1}^{k} a_j t^{k-j} \partial_t^{m-j} + \sum_{l=0}^{m} b_l(t) t^{d(l)} \partial_t^l \ ,$$

where $0 \le k \le m$, $a_j \in \mathbb{C}$, and $b_l \in C^\infty(-T_0, T_0)$. Namely, P is an operator with a regular singularity at $t = 0$ having C^∞ coefficients.

Put

$$\mathcal{C}[P](\lambda) = \mathcal{C}(\lambda) := \{t^{-\lambda+\omega} P(t^\lambda)\}|_{t=0} = (\lambda)_m + \sum_{j=1}^{k} a_j (\lambda)_{m-j} \in \mathbb{C}[\lambda] \ ,$$

which is called *the indicial polynomial* of P. A root of $\mathcal{C}(\lambda) = 0$ is called a *characteristic exponent(index)* of P. We can decompose $\mathcal{C}$ as

$$\mathcal{C}(\lambda) = (\lambda)_\omega \widetilde{\mathcal{C}}(\lambda - \omega) \ ,$$

where $\widetilde{\mathcal{C}}[P](\lambda) = \widetilde{\mathcal{C}}(\lambda) := (\lambda)_k + \sum_{j=1}^{k} a_j (\lambda)_{k-j} \in \mathbb{C}[\lambda]$.

Let $\widetilde{\mathcal{C}}(\lambda) = \prod_{l=1}^{d} (\lambda - \lambda_l)^{r_l}$, where $d \in \mathbb{N}$, $r_l \ge 1$, and $(\lambda_1, \dots, \lambda_d)$ are distinct.

2.1 Formal Solutions

First, we consider the solutions in the space $\mathcal{F}$ of formal series of the form

$$u(t) = t^\rho \sum_{j=0}^{\infty} t^j \sum_{\nu=0}^{q_j} a_{j,\nu} (\log t)^\nu \ ,$$

where $\rho \in \mathbb{C}$, $q_j \in \mathbb{N}$, and $a_{j,\nu} \in \mathbb{C}$. Note that we have only to consider $t^\omega P$ instead of P in this space, since $Pu = 0$ if and only if $t^\omega Pu = 0$. Thus, we assume $\omega = 0$ ($k = m$) without loss of generality.

Theorem 1. *Assume $k = m$ and put $\operatorname{Ker}_{\mathcal{F}} P := \{\, u \in \mathcal{F} \mid Pu = 0 \,\}$. For every l with $1 \le l \le d$ and for every p with $1 \le p \le r_l$, there exists $v_{l,p} = t^{\lambda_l} (\log t)^{p-1} + \sum_{j=1}^{\infty} t^{\lambda_l + j} \sum_{\nu=0}^{q_j} a_{j,\nu} (\log t)^\nu \in \operatorname{Ker}_{\mathcal{F}} P$, where $a_{j,\nu} \in \mathbb{C}$. Further, these $m(= r_1 + \ldots + r_d)$ solutions make a base of $\operatorname{Ker}_{\mathcal{F}} P$. Especially, there holds $\dim \operatorname{Ker}_{\mathcal{F}} P = m$.*

Remark 2. (1) If the coefficients of P are holomorphic in a neighborhood of 0, then this formal solution converges in $\mathcal{O}(\mathcal{R}(B \setminus \{0\}))$ for some domain B including 0, where $\mathcal{R}(V)$ denotes the universal covering of V.
(2) If $r_l = 1$ for every l and if $\{\, \lambda_l \,\}$ do not differ by integer, then we can take $q_j = 0$ for every $j \in \mathbb{N}$, that is, the solutions never include the terms with $\log t$.

2.2 Solutions in $\mathcal{D}'_+$

Next, we consider the solutions of $Pu = 0$ in $\mathcal{D}'_+(-T_0, T_0)$. In this case, we can not reduce to the case where $\omega = 0$, since $Pu = 0$ is not equivalent to $t^\omega Pu = 0$.

Put

$$G(z) = G(z; t) := \frac{t_+^z}{\Gamma(z+1)} \in \mathcal{D}'_+(\mathbb{R}; \mathcal{O}(\mathbb{C})) \ , \tag{2}$$

$$G^{(j)}(z) := \partial_z^j (G(z)) \in \mathcal{D}'_+(\mathbb{R}; \mathcal{O}(\mathbb{C})) \ . \tag{3}$$

Note that $\partial_t^h G(z) = G(z - h)$ ($h \in \mathbb{N}$) and that $G(-d) = \partial_t^d (G(0)) = \delta^{(d-1)}(t)$ for $d = 1, 2, \ldots$.

Theorem 3. *Put $\operatorname{Ker}_{\mathcal{D}'_+} P := \{\, u \in \mathcal{D}'_+(-T_0, T_0) \mid Pu = 0 \,\}$. For every l with $1 \le l \le d$ and for every p with $1 \le p \le r_l$, there exists $u_{l,p} \in \operatorname{Ker}_{\mathcal{D}'_+} P$ satisfying $u_{l,p} \sim G^{(p-1)}(\lambda_l + \omega) + \sum_{j=1}^{\infty} \sum_{\nu=0}^{q_j} a_{j,\nu} G^{(\nu)}(\lambda_l + \omega + j)$. Here, $\sim$ means that for every $N \in \mathbb{N}$, there holds $u_{l,p} - G^{(p-1)}(\lambda_l + \omega) - \sum_{j=1}^{N} \sum_{\nu=0}^{q_j} a_{j,\nu} G^{(\nu)}(\lambda_l + \omega + j) \in C_+^{\lceil \operatorname{Re} \lambda_l \rceil + \omega + N}(-T_0, T_0)$, where $\lceil a \rceil$ denotes the smallest integer M satisfying $M \ge a \in \mathbb{R}$.*

Further, these $k(= r_1 + \ldots + r_d)$ solutions make a base of $\operatorname{Ker}_{\mathcal{D}'_+} P$. Especially, there holds $\dim \operatorname{Ker}_{\mathcal{D}'_+} P = k$. (Cf. Similarly, we have $\dim \operatorname{Ker}_{\mathcal{D}'} P = m + k$.)

Example 4. (1) Consider $P = (\vartheta - d + 1)\partial_t = \partial_t(\vartheta - d)$, where $d \in \mathbb{N}$ and $d \geq 1$. We have $m = 2$, $k = 1$, $\omega = 1$, and $\mathcal{C}(\lambda) = \lambda(\lambda - d)$. We have $\mathrm{Ker}_{\mathcal{F}} P = \mathrm{Ker}_{\mathcal{F}}(tP) = \mathrm{Span}\{1, t^d\}$, $\mathrm{Ker}_{\mathcal{D}'_+} P = \mathrm{Span}\{t_+^d\}$, $\mathrm{Ker}_{\mathcal{D}'} P = \mathrm{Span}\{t_+^d, t^d, 1\}$.
(2) Consider $P = (\vartheta + d + 1)\partial_t = \partial_t(\vartheta + d)$, where $d \in \mathbb{N}$ and $d \geq 1$. We have $m = 2$, $k = 1$, $\omega = 1$, and $\mathcal{C}(\lambda) = \lambda(\lambda + d)$. We have $\mathrm{Ker}_{\mathcal{F}} P = \mathrm{Ker}_{\mathcal{F}}(tP) = \mathrm{Span}\{1, t^{-d}\}$, $\mathrm{Ker}_{\mathcal{D}'_+} P = \mathrm{Span}\{\delta^{(d-1)}\}$, $\mathrm{Ker}_{\mathcal{D}'} P = \mathrm{Span}\{\delta^{(d-1)}, 1, (t + i0)^{-d}\}$.
(3) Consider $P = (\vartheta - d + 1)^2\partial_t = \partial_t(\vartheta - d)^2$, where $d \in \mathbb{N}$ and $d \geq 1$. We have $m = 3$, $k = 2$, $\omega = 1$, and $\mathcal{C}(\lambda) = \lambda(\lambda - d)^2$. We have $\mathrm{Ker}_{\mathcal{F}} P = \mathrm{Ker}_{\mathcal{F}}(tP) = \mathrm{Span}\{1, t^d, t^d \log t\}$, $\mathrm{Ker}_{\mathcal{D}'_+} P = \mathrm{Span}\{t_+^d, t_+^d \log t_+\}$, $\mathrm{Ker}_{\mathcal{D}'} P = \mathrm{Span}\{t_+^d, t_+^d \log t_+, 1, t^d, t^d \log(t + i0)\}$.
(4) Consider $P = (\vartheta + 1)\partial_t = \partial_t\vartheta$. We have $m = 2$, $k = 1$, $\omega = 1$, and $\mathcal{C}(\lambda) = \lambda^2$. $\mathrm{Ker}_{\mathcal{F}} P = \mathrm{Ker}_{\mathcal{F}}(tP) = \mathrm{Span}\{1, \log t\}$, $\mathrm{Ker}_{\mathcal{D}'_+} P = \mathrm{Span}\{t_+^0\}$, $\mathrm{Ker}_{\mathcal{D}'} P = \mathrm{Span}\{t_+^0, 1, \log(t + i0)\}$.

Remark 5. As in §2.1, if $r_l = 1$ for every l and if $\{ \lambda_l \}$ do not differ by integer, then we can take $q_j = 0$ for every j.

2.3 Rough Statement of our Result

We want to prove a similar fact for Fuchsian partial differential equations. We shall show that

$$(\mathcal{O}_0)^k \cong (\mathrm{Ker}_{\mathcal{D}'_+} P)_{(0,0)} \quad , \tag{4}$$

constructing the isomorphism (invertible linear map) rather concretely, where $(\ldots)_0$ and $(\ldots)_{(0,0)}$ denote the spaces of all germs. Namely, $\mathcal{O}_0 := \operatorname*{indlim}_{0 \in \Omega \subset \mathbb{C}^n} \mathcal{O}(\Omega)$ and $(\mathrm{Ker}_{\mathcal{D}'_+} P)_{(0,0)} := \operatorname*{indlim}_{T>0; 0 \in \Omega \subset \mathbb{C}^n} \mathrm{Ker}_{\mathcal{D}'_+(-T,T;\mathcal{O}(\Omega))} P$.

We shall state in a little more detail. Consider an operator (1) with $a_j \in \mathcal{O}(\Omega_0)$ and $b_{l,\alpha} \in C^\infty(-T_0, T_0; \mathcal{O}(\Omega_0))$, where $T_0 > 0$ and Ω_0 is a domain in $\mathbb{C}^n$ including 0. Define

$$\mathcal{C}(x; \lambda) := (\lambda)_m + \sum_{j=1}^{k} a_j(x)(\lambda)_{m-j} = \{t^{-\lambda+\omega} P(t^\lambda)\}|_{t=0} \quad ,$$

which is also called *the indicial polynomial* of P, and a root λ of $\mathcal{C}(x; \lambda) = 0$ is called a *characteristic exponent* of P. The indicial polynomial can be decomposed as

$$\mathcal{C}(x; \lambda) = (\lambda)_\omega \widetilde{\mathcal{C}}(x; \lambda - \omega) \quad ,$$

where

$$\widetilde{\mathcal{C}}(x; \lambda) := (\lambda)_k + \sum_{j=1}^{k} a_j(x)(\lambda)_{k-j} \quad .$$

Let $\widetilde{\mathcal{C}}(0; \lambda) = \prod_{l=1}^{d} (\lambda - \lambda_l)^{r_l}$, where $d \in \mathbb{N}$, $r_l \geq 1$, and $(\lambda_1, \ldots, \lambda_d)$ are distinct.

Rough Statement of our result is the following.

Theorem 6. *There exist $T > 0$ and a subdomain Ω of Ω_0 including 0 such that for every l with $1 \le l \le d$ and for every p with $1 \le p \le r_l$, there exists a continuous linear map $u_{l,p}$ from $\mathcal{O}(\Omega_0)$ to $\mathcal{D}'_+(-T, T; \mathcal{O}(\Omega))$ satisfying the following.*

For every $a \in \mathcal{O}(\Omega_0)$, there holds $P(u_{l,p}[a]) = 0$ and

$$u_{l,p}[a]|_{x=0} \sim a(0)G^{(p-1)}(\lambda_l + \omega) + \sum_{h=1}^{\infty}\sum_{\nu=0}^{q_h} a_{h,\nu}G^{(\nu)}(\lambda_l + \omega + h)$$

for some $q_h \in \mathbb{N}$ and $a_{h,\nu} \in \mathbb{C}$.

Conversely, every solution $u \in \mathcal{D}'_+(-T_0, T_0; \mathcal{O}(\Omega_0))$ of $Pu = 0$ is represented by $\sum_{l,p} u_{l,p}[a_{l,p}]$ for some $a_{l,p} \in \mathcal{O}(\Omega)$ $(1 \le l \le d,\ 1 \le p \le r_l)$ in a neighborhood of $(0, 0)$.

Further, if there exists (l, p) such that $a_{l,p} \not\equiv 0$, then $(0, 0) \in \operatorname{supp} u$, that is, u is a null-solution.

In the result by T.Mandai([6]) stated in Introduction, he constructed a solution corresponding to the solution $u_{l,0}$ for a root λ_l satisfying that $\widetilde{\mathcal{C}}(0; \lambda_l + j) \ne 0$ for $j = 1, 2, \ldots$.

The major difficulty of the proof is the fact that a root of $\widetilde{\mathcal{C}}(x; \lambda) = 0$ is not necessarily holomorhic in x, and this becomes a bigger difficulty when there exist two roots with integer difference, as suggested from the case of ordinary differential equations.

3 Preliminaries

In this section, we give some lemmas and propositions needed for the proof of our result.

For each $1 \le l \le d$, we take a domain D_l in $\mathbb{C}$ enclosed by a simple closed curve Γ_l such that the following three conditions hold.
(a) $\lambda_l \in D_l$ $(1 \le l \le d)$.
(b) $\overline{D_l} \cap \overline{D_{l'}} = \emptyset$ if $l \ne l'$.
(c) if $j \in \mathbb{N}$ and if $\lambda_{l'} - j \in \overline{D_l}$, then $\lambda_{l'} - j = \lambda_l$.
(This is equivalent to "$\{ \lambda_{l'} - j \in \mathbb{C} \mid 1 \le l' \le d, j \in \mathbb{N} \} \cap \overline{D_l} = \{ \lambda_l \}$ for every l". Also these are equivalent to "$\widetilde{\mathcal{C}}(0; \lambda + j) \ne 0$ for every $\lambda \in \bigcup_{l=1}^{d}(\overline{D_l} \setminus \{ \lambda_l \})$ and for every $j \in \mathbb{N}$".) Note that $\{ \lambda_{l'} - j \in \mathbb{C} \mid 1 \le l' \le d, j \in \mathbb{N} \}$ is a discrete set and hence we can take such Γ_l.

There exist a doman Ω in $\mathbb{C}^n$ including 0, and monic polynomials $E_l(x; \lambda) \in \mathcal{O}(\Omega)[\lambda]$ $(1 \le l \le d)$ such that
(d) $\widetilde{\mathcal{C}}(x; \lambda) = \prod_{l=1}^{d} E_l(x; \lambda)$,
(e) $E_l(0; \lambda) = (\lambda - \lambda_l)^{r_l}$ $(1 \le l \le d)$,
(f) for $1 \le l \le d$, if $E_l(x; \lambda) = 0$ and $x \in \Omega$, then $\lambda \in D_l$,
(g) $\widetilde{\mathcal{C}}(x; \lambda + j) \ne 0$ for every $x \in \Omega$, every $\lambda \in \bigcup_{l=1}^{d} \Gamma_l$, and every $j \in \mathbb{N}$.

Further, by reducing D_l and Ω if necessary, we can take $\epsilon \ge 0$ and $L_l \in \mathbb{Z}$ $(1 \le l \le d)$ such that

(h) If $\widetilde{\mathcal{C}}(x;\lambda) = 0$ and if $x \in \Omega$, then $\mathrm{Re}\lambda - \epsilon \notin \mathbb{Z}$. Further, $\overline{D_l} \subset \{\, \lambda \in \mathbb{C} \mid L_l + \epsilon < \mathrm{Re}\lambda < L_l + \epsilon + 1 \,\}$.

Definition 7. For $1 \le l \le d$, $j \in \mathbb{N}$, and for $\phi \in \mathcal{O}(\Omega \times \Gamma_l)$, put

$$\mathcal{H}_{l,j}[\phi](t,x) := \frac{1}{2\pi i}\int_{\Gamma_l} \frac{\phi(x;\zeta)}{E_l(x;\zeta)} G(\zeta + j; t)\, d\zeta \quad \in \mathcal{D}'_+(\mathbb{R}; \mathcal{O}(\Omega)) \ .$$

Also for $1 \le p \le r_l$, put

$$w_{l,p}(t,x) := \frac{(r_l - p)!}{r_l!}\mathcal{H}_{l,\omega}[\partial_\zeta^p E_l](t,x) \ . \tag{5}$$

Note that $w_{l,p}(t,0) = \dfrac{1}{(p-1)!}G^{(p-1)}(\lambda_l + \omega)$.

Remark 8. If we fix $x = x_0$, then $w_{l,p}(t,x_0) = \sum_{j,k:\text{finite}} c_{j,k} G^{(k)}(\mu_j + \omega)$, where $c_{j,k} \in \mathbb{C}$, and $\{\mu_j\}$ are the roots of $\widetilde{\mathcal{C}}(x_0;\lambda) = 0$.

Proposition 9. (1) *$\{w_{l,p}(\cdot,x)\}_{l,p}$ is a base of* $\mathrm{Ker}_{\mathcal{D}'_+}\, \widetilde{\mathcal{C}}(x;\vartheta)\partial_t^\omega$ *for every fixed $x \in \Omega$.*
(2) *If $u \in \mathcal{D}'_+(\mathbb{R};\mathcal{O}(\Omega))$ satisfies $\widetilde{\mathcal{C}}(x;\vartheta)\partial_t^\omega u = 0$ for every $x \in \Omega$, then $u = \sum_{l,p} a_{l,p}(x) w_{l,p}(t,x)$ for some $a_{l,p} \in \mathcal{O}(\Omega)$. The point is the holomorphy of $a_{l,p}$.*

Example 10. Consider $P = \vartheta^2 - x = E_1(x;\vartheta)$, where $d = 1 (= l)$, $r_1 = 2$, and $\omega = 0$. We have

$$w_{1,1} = \frac{1}{2}\mathcal{H}_{1,0}[2\zeta] = \frac{1}{2}\frac{1}{2\pi i}\int_{\Gamma_1} \frac{2\zeta}{\zeta^2 - x} G(\zeta;t)\, d\zeta = \frac{1}{2}\{G(\sqrt{x};t) + G(-\sqrt{x};t)\},$$
$$w_{1,2} = \frac{1}{2}\mathcal{H}_{1,0}[2] = \frac{1}{2}\frac{1}{2\pi i}\int_{\Gamma_1} \frac{2}{\zeta^2 - x} G(\zeta;t)\, d\zeta = \frac{G(\sqrt{x};t) - G(-\sqrt{x};t)}{2\sqrt{x}} \ .$$

Proposition 11. (1) $\partial_t^h \mathcal{H}_{l,j}[\phi] = \mathcal{H}_{l,j-h}[\phi]$.
(2) $t^h \mathcal{H}_{l,j}[\phi] = \mathcal{H}_{l,j+h}[(\zeta + j + h)_h \phi]$.
(3) *For $F(x;\lambda) \in \mathcal{O}(\Omega)[\lambda]$, there holds $F(x;\vartheta)\mathcal{H}_{l,j}[\phi] = \mathcal{H}_{l,j}[F(x;\zeta + j)\phi]$.*
(4) *$\partial_{x_\nu}\mathcal{H}_{l,j}[\phi] = \mathcal{H}_{l,j}[L_\nu(\phi)]$, where*

$$L_\nu(\phi)(x;\zeta) := (\partial_{x_\nu}\phi)(x;\zeta) - \frac{(\partial_{x_\nu}E_l)(x;\zeta)}{E_l(x;\zeta)}\phi(x;\zeta) \ .$$

Proposition 12. *$\mathcal{H}_{l,j}[\phi] \in C_+^{j+L_l}(\mathbb{R};\mathcal{O}(\Omega))$.*

For $1 \le l \le d$ and $1 \le p \le r_l$, we can construct an asymptotic solution of $Pu = 0$ in the form of

$$u = a(x) w_{l,p}(t,x) + \sum_{h=1}^{\infty} \mathcal{H}_{l,\omega+h}[\mathcal{S}_h(a)](t,x) \ ,$$

where $\mathcal{S}_h = \mathcal{S}_{l,p,h}$ is a continuous linear map from $\mathcal{O}(\Omega)$ to $\mathcal{O}(\Omega \times \Gamma_l)$ of the form

$$\mathcal{S}_h(a)(x;\zeta) = \sum_{|\alpha| \le mh} s_{h,\alpha}(x;\zeta)\partial_x^\alpha a(x) \ ,$$

where

$$s_{h,\alpha} = s_{l,p,h,\alpha} \in \frac{1}{\prod_{j=0}^{h} \widetilde{\mathcal{C}}(x;\zeta+j)^{m_h}} \times \mathcal{O}(\Omega \times \overline{D_l}) \ ,$$

for some $m_h \in \mathbb{N}$. Further, there exists $q_h \in \mathbb{N}$ and $a_{h,\nu} \in \mathbb{C}$ $(h \ge 1; 0 \le \nu \le q_h)$ such that

$$u(t,0) \sim a(0)\frac{1}{(p-1)!}G^{(p-1)}(\lambda_l + \omega) + \sum_{h=1}^{\infty}\sum_{\nu=0}^{q_h} a_{h,\nu}G^{(\nu)}(\lambda_l + \omega + h) \ .$$

4 Detailed Statement of our Result

Now, we can state our result in a full detail.

Let Ω be a subdomain of Ω_0 including 0 and $T \in (0, T_0)$.

Theorem 13. *There exist $T' \in (0,T)$ and a subdomain Ω' of Ω including 0 such that for every l with $1 \le l \le d$ and for every p with $1 \le p \le r_l$, the following holds: There exists a continuous linear map $u_{l,p}$ from $\mathcal{O}(\Omega)$ to $C_+^{L_l+\omega}(-T',T';\mathcal{O}(\Omega'))$ such that for every $a \in \mathcal{O}(\Omega)$, there holds*

(i) $P(u_{l,p}[a]) = 0$.
(ii) $u_{l,p}[a](t,x) \sim a(x)w_{l,p}(t,x) + \sum_{h=1}^{\infty} \mathcal{H}_{l,\omega+h}[\mathcal{S}_h(a)](t,x)$,

Theorem 14. *If $u \in \mathcal{D}'_+(-T,T;\mathcal{O}(\Omega))$ satisfies $Pu = 0$, then there exists a subdomain Ω' of Ω including 0 and there exists a unique $a_{l,p} \in \mathcal{O}(\Omega')$ such that $u = \sum_{l,p} u_{l,p}[a_{l,p}]$ in a neighborhood of $(0,0)$. Further, if there exists (l,p) such that $a_{l,p} \not\equiv 0$, then $(0,0) \in \operatorname{supp} u$, that is, u is a distribution null-solution for P.*

These two theorems imply that $(\mathcal{O}_0)^k \cong (\mathrm{Ker}_{\mathcal{D}'_+} P)_{(0,0)}$, as we already stated. We can also show that $(\mathcal{O}_0)^{m+k} \cong (\mathrm{Ker}_{\mathcal{D}'} P)_{(0,0)}$, similarly.

We can prove Theorem 13 by realizing the asymptotic solution constructed in the previous section.

We shall give a sketch of a proof of Theorem 14 in the next section.

5 Sketch of the Proof of Theorem 14

First, we give a sketch of the uniqueness of $a_{l,p}$.

We have taken $\epsilon \ge 0$ such that if $\widetilde{\mathcal{C}}(x;\lambda) = 0$ $(x \in \Omega)$, then $\mathrm{Re}\lambda - \epsilon \notin \mathbb{Z}$. We have also taken $L_l \in \mathbb{Z}$ such that if $x \in \Omega$ and if $E_l(x;\lambda) = 0$, then $L_l + \epsilon < \mathrm{Re}\lambda < L_l + \epsilon + 1$.

Definition 15. For $L \in \mathbb{Z}$, we put

$$W_L^{(N)}(-T,T;X) := \begin{cases} \biguplus_{s=0}^{N} \vartheta^s \partial_t^{|L|}(t^\epsilon \times C_+^0(-T,T;X)) \ (L \le 0) \\ \biguplus_{s=0}^{N} \vartheta^s (t^{L+\epsilon} \times C_+^0(-T,T;X)) \ \ (L \ge 0) \end{cases} .$$

Note that $W_L^{(N)}(-T,T;\mathcal{O}(\Omega)) \subset W_{L-1}^{(N)}(-T,T;\mathcal{O}(\Omega))$ and $W_L^{(N)}(-T,T;\mathcal{O}(\Omega)) \subset W_L^{(N+1)}(-T,T;\mathcal{O}(\Omega))$.

Take $\chi(t) \in \mathcal{D}(-T,T)$ with $\chi(t)=1$ near $t=0$. Then, we have the following.

$$\mathcal{H}_{l,j}[\phi] \in W_{L_l+j}^{(0)} \ , \tag{6}$$

$$\text{if } v \in W_L^{(N)}, \text{ then } \langle v, \chi(t)e^{-t/\rho}\rangle_t = o(\rho^{L+\epsilon+1}) \ (\rho \to +0) \ . \tag{7}$$

If we fix an arbitrary x, then there exists $a_{j,k} \in \mathbb{C}$ such that

$$\begin{aligned} \langle w_{l,p}, \chi(t)e^{-t/\rho}\rangle_t &= \textstyle\sum_{j,k:\text{finite}} a_{j,k}\rho^{\mu_j+\omega+1}(\log\rho)^k + o(\rho^\infty) \\ &(= o(\rho^{L_l+\omega+\epsilon+1})) \ (\rho \to +0) \ , \end{aligned} \tag{8}$$

where $\{\mu_j\}$ are the roots of $\widetilde{\mathcal{C}}(x;\lambda)=0$, since $\langle G^{(\nu)}(\lambda), e^{-t/\rho}\rangle_t = \rho^{\lambda+1}(\log\rho)^\nu$.

From these, we can show that if $\sum_{l,p} u_{l,p}[a_{l,p}] = 0$, then $a_{l,p}=0$ for every (l,p).

Next, we give a sketch of the existence of $a_{l,p}$ for a given solution u.

Proposition 16. (i) $\mathcal{D}'_+(-\widetilde{T},\widetilde{T};\mathcal{O}(\Omega)) \subset \bigcup_{L\in\mathbb{Z}} W_L^{(0)}(-T,T;\mathcal{O}(\Omega))$, *if* $\widetilde{T} > T$.

(ii) $t \times W_L^{(N)}(-T,T;\mathcal{O}(\Omega)) \subset \begin{cases} W_{L+1}^{(N+1)}(-T,T;\mathcal{O}(\Omega)) \ (L \le -1) \\ W_{L+1}^{(N)}(-T,T;\mathcal{O}(\Omega)) \ \ (L \ge 0) \end{cases}$,

$\partial_t(W_L^{(N)}(-T,T;\mathcal{O}(\Omega))) \subset \begin{cases} W_{L-1}^{(N)}(-T,T;\mathcal{O}(\Omega)) \ \ (L \le 0) \\ W_{L-1}^{(N+1)}(-T,T;\mathcal{O}(\Omega)) \ (L \ge 1) \end{cases}$,

$\vartheta(W_L^{(N)}(-T,T;\mathcal{O}(\Omega))) \subset W_L^{(N+1)}(-T,T;\mathcal{O}(\Omega))$.

(iii) *For sufficiently large L, there holds*

$$\mathrm{Ker}_{\mathcal{D}'_+(-T,T;\mathcal{O}(\Omega))} P \cap W_L^{(N)}(-T,T;\mathcal{O}(\Omega)) = \{0\}$$

for every $N \in \mathbb{N}$.

(iv) *For every* $g \in W_L^{(N)}(-T,T;\mathcal{O}(\Omega))$, *there exists* $v \in W_{L+\omega}^{(N)}(-T,T;\mathcal{O}(\Omega))$ *such that* $\widetilde{\mathcal{C}}(x;\vartheta)\partial_t^\omega v = g$.

(v) *For* $1 \le l \le d$ *and* $1 \le p \le r_l$, *there holds* $w_{l,p} \in W_{L_l+\omega}^{(0)}(-T,T;\mathcal{O}(\Omega))$. *Further, for every* $L \in \mathbb{Z}$ *and every* $N \in \mathbb{N}$, *there holds*

$$\begin{aligned} &\mathrm{Ker}_{\mathcal{D}'_+(-T,T;\mathcal{O}(\Omega))} \widetilde{\mathcal{C}}(x;\vartheta)\partial_t^\omega \cap W_L^{(N)}(-T,T;\mathcal{O}(\Omega)) \\ &\qquad = \mathrm{Span}\{w_{l,p} \mid 1 \le l \le d, \ L_l+\omega \ge L, \ 1 \le p \le r_l\} \ . \end{aligned}$$

From these, we can show the existence of $a_{l,p}$ as follows.

Let $u \in \mathcal{D}'_+(-T,T;\mathcal{O}(\Omega))$ and $Pu = 0$. By (i) and by reducing T, there exists $L \in \mathbb{Z}$ such that $u \in W_L^{(0)}(-T,T;\mathcal{O}(\Omega))$. Putting $P = \widetilde{\mathcal{C}}(x;\vartheta)\partial_t^\omega + R$, we have $\widetilde{\mathcal{C}}(x;\vartheta)\partial_t^\omega u = -Ru \in W_{L-\omega+1}^{(m)}(-T,T;\mathcal{O}(\Omega))$ by (ii). By (iv), there exists $v \in W_{L+1}^{(m)}(-T,T;\mathcal{O}(\Omega))$ such that $u - v \in \mathrm{Ker}_{\mathcal{D}'_+(-T,T;\mathcal{O}(\Omega))}\, \widetilde{\mathcal{C}}(x;\vartheta)\partial_t^\omega$. Since $u - v \in W_L^{(m)}(-T,T;\mathcal{O}(\Omega))$, there exists $a_{l,p}[0] \in \mathcal{O}(\Omega)$ ($1 \le l \le d$, $L_l + \omega \ge L$, $1 \le p \le r_l$) such that

$$u - v = \sum_{1\le l\le d,\ L_l+\omega\ge L,\ 1\le p\le r_l} a_{l,p}[0] w_{l,p} \ ,$$

by (v). Put $u[1] := u - \sum_{l,p} u_{l,p}[a_{l,p}[0]]$, then wa have $P(u[1]) = 0$ and we can show $u[1] \in W_{L+1}^{(m)}(-T,T;\mathcal{O}(\Omega))$.

Similarly, we get $u[2] := u[1] - \sum_{l,p} u_{l,p}[a_{l,p}[1]] \in \mathrm{Ker}_{\mathcal{D}'_+(-T,T;\mathcal{O}(\Omega))} P \cap W_{L+2}^{(2m)}(-T,T;\mathcal{O}(\Omega))$, for some $a_{l,p}[1] \in \mathcal{O}(\Omega)$ by reducing Ω and T. By repeating this argument, we get $u[N] \in \mathrm{Ker}_{\mathcal{D}'_+(-T,T;\mathcal{O}(\Omega))} P \cap W_{L+N}^{(Nm)}(-T,T;\mathcal{O}(\Omega))$ that can be written as $u[N] = u - \sum_{l,p} u_{l,p}[a_{l,p}]$ for some $a_{l,p} \in \mathcal{O}(\Omega)$. By (iii), we get $u[N] = 0$, and hence u can be written as $u = \sum_{l,p} u_{l,p}[a_{l,p}]$ for some $a_{l,p} \in \mathcal{O}(\Omega)$.

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