WAVELET BASES FOR MICROLOCAL FILTERING AND THE SAMPLING THEOREM IN $L_p(\mathbb{R}^n)$

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Dedicated to Professor Mitsuru IKAWA on his sixtieth birthday

Abstract. An orthonormal wavelet basis in $L_2(\mathbb{R}^n)$ used for microlocal filters, which decompose signals into microlocal contents, is shown to be a "stepwise" unconditional basis in $L_p(\mathbb{R}^n)$ $(1 < p < \infty)$. Other related spaces are also treated. As part of the proof, an elementary proof of the L_p version of the sampling theorem with unconditional convergence is given. Finally, an application is given to the expression of some distributions as sums of boundary values of holomorphic functions.

1. Introduction

The extraordinary development of wavelets in recent years has made them present in a large part of our high-technology world ([7], [13]). Wavelets are being incorporated in engineering standards for image and audio signal compression. The first standard based on wavelets is "wavelet scalar quantization" adopted by the U.S. Federal Bureau of Investigation (FBI) in 1997 to encode fingerprints. The new still-image compression standard known as JPEG2000 includes a wavelet option, and MPEG-4, the next video compression standard, will be entirely wavelet-based. Developments in wavelets have influenced a large number of pure and applied mathematicians, and scientists in such disparate fields as numerical analysis, computer vision, human vision, turbulence, statistics, physics, and medicine.

Most systems in engineering are modeled as analog, but most of their computational engines are digital. Transforming from analog to digital is straightforward by what we call "sampling". Regaining the original signal from sampled data or assessing the information lost in the sampling

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process are fundamental questions in sampling theory ([14]). The classical sampling theorem, usually associated with the names of E. T. Whittaker, V. A. Kotel'nikov and C. E. Shannon, provides the theoretical foundation for communications systems. We say that a function (or distribution) f is *band-limited* to I if supp $\widehat{f} \subset I$, where $\widehat{f}$ denotes the Fourier transform of f:

$$\widehat{f}(\xi) = f^{\wedge}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx \ . \tag{1.1}$$

The classical sampling theorem states that functions band-limited to $[-\sigma\pi, \sigma\pi]$ can be reconstructed from uniform samples $\{f(k/\sigma)\}_{k\in\mathbb{Z}}$ by

$$f(t) = \sum_{k\in\mathbb{Z}} f(k/\sigma) \frac{\sin \pi(\sigma t - k)}{\pi(\sigma t - k)} \ . \tag{1.2}$$

The classical sampling theorem has been generalized in various directions. One such generalization given by F. Gensun [4] extends the function space, which contains band-limited functions, from $L_2(\mathbb{R})$ to $L_p(\mathbb{R})$, $1 < p < \infty$.

Wavelets have been developed as one of several tools for time-frequency analysis, which could be called "local Fourier analysis". Another "local Fourier analysis" named *microlocal analysis* has been developed extensively in the theory of hyperfunctions introduced by M. Sato [17]. Hyperfunctions, which are a very wide generalization of functions, can be considered as sums of formal boundary values of holomorphic functions defined in infinitesimal wedges (Figure 1). They are powerful tools in several applications; for

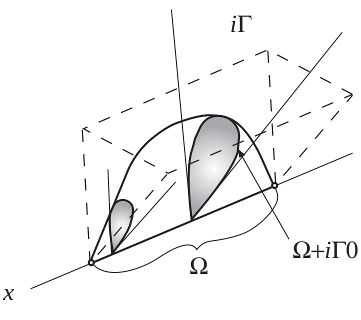

FIGURE 1. An infinitesimal wedge $\Omega + i\Gamma 0$.

example, vortex sheets in two-dimensional fluid dynamics are a realization of hyperfunctions of one variable ([8]). Microlocal analysis deals with the direction along which a hyperfunction can be extended analytically. In other words, it decomposes the "singularity" into microlocal directions. Microlocal analysis plays an important role in the theory of hyperfunctions, partial differential operators, and many other areas. In this theory, for example, one can consider the product of hyperfunctions and discuss the partial regularity of hyperfunctions with respect to any independent variable.

The article [2] constructed orthonormal wavelets in $L_2(\mathbb{R}^n)$ applicable to microlocal analysis. Since microlocal decomposition can be done numerically by a filtering algorithm using those orthonormal wavelets, such wavelets are called *microlocal filters*. The orthonormal wavelet basis enables us to obtain information on the microlocal contents of signals or functions.

The main purpose of this article is to show that the orthonormal wavelet bases in $L_2(\mathbb{R}^n)$ are "stepwise unconditional bases" in $L_p(\mathbb{R}^n)$, $1 < p < \infty$. Related spaces are also treated. To prove these results, the L_p version of the sampling theorem is used. Its elementary proof will be given and the unconditionality of the convergence of (1.2), which was not stated explicitly in [4], will be shown.

In the next section, we review the results of [2]. In Section 3, the function spaces we consider are introduced. We give the precise statement of our main results in Section 4. After giving some preliminaries in Section 5, we give the L_p version of the sampling theorem in Section 6. Sections 7 and 8 are devoted to the proofs of the main theorems. In the final section, we give an application to the expression of some distributions as sums of boundary values of holomorphic functions.

2. Microlocal Filters

In this section, we will give a brief overview of [2].

Notation:

- $\mathbb{Z} := \{\text{Integers}\}$, $\mathbb{Z}_+ := \{ n \in \mathbb{Z} \,:\, n \geq 0 \}$,
  $\mathbb{N} := \{ n \in \mathbb{Z} \,:\, n > 0 \}$, $\mathbb{R} := \{\text{Real numbers}\}$,
  $\mathbb{R}_\pm := \{ t \in \mathbb{R} \,:\, \pm t > 0 \}$, $\mathbb{R}_\star := \{ t \in \mathbb{R} \,:\, t \neq 0 \}$,
  $\mathbb{R}^n_\star := (\mathbb{R}_\star)^n$, $\mathbb{C} := \{\text{Complex numbers}\}$,
  $\mathbb{S}^{n-1}$ denotes the $(n-1)$-dimensional unit sphere.
- For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, the inner product of x and y is denoted by $x \cdot y := \sum_{\nu=1}^n x_\nu y_\nu$. The L_2 inner product is denoted by $\langle f, g \rangle := \displaystyle\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx$.
- $\widehat{f}(\xi) = f^\wedge(\xi) := \displaystyle\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$ (Fourier transform of f). As is well-known, this is an isomorphism on each of the three spaces $\mathscr{S}(\mathbb{R}^n) \subset L_2(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$, where $\mathscr{S}(\mathbb{R}^n)$ is the space of rapidly decreasing C^∞ functions, and $\mathscr{S}'(\mathbb{R}^n)$ is its dual, that is, the space of

tempered distributions. We also use the variables $\omega = (\omega_1, \dots, \omega_n)$ with $\xi = 2\pi\omega$.

For $X \subset \mathscr{S}'(\mathbb{R}^n)$, set $\widehat{X} := \{\, \widehat{f} \ : \ f \in X \,\}$. If $f \in X$ implies $\tilde{f} \in X$, where $\tilde{f}(x) := f(-x)$, then $\widehat{X} = \{\, f \in \mathscr{S}'(\mathbb{R}^n) \ : \ \widehat{f} \in X \,\}$.

- $g^{\vee}(x) := \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)\, d\xi$ (inverse Fourier transform of g).

2.1. **Orthonormal Wavelets.** For $f \in L_2(\mathbb{R}^n)$, let $f_{jk}(x)$ denote the scaled and shifted function

$$f_{jk}(x) = 2^{nj/2} f(2^j x - k), \qquad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^n \ . \tag{2.1}$$

Definition 2.1. Let D be a finite index set. A system $\{(\psi_\delta)_{jk}\}_{\delta\in D, j\in\mathbb{Z}, k\in\mathbb{Z}^n} \subset L_2(\mathbb{R}^n)$ is called an *orthonormal wavelet basis* and a system $\{\psi_\delta\}_{\delta\in D}$ is called a *system of orthonormal wavelet functions*, if the system $\{(\psi_\delta)_{jk}\}_{\delta\in D, j\in\mathbb{Z}, k\in\mathbb{Z}^n}$ is an orthonormal basis for $L_2(\mathbb{R}^n)$.

2.2. **Microlocal Analysis.** Our approach to microlocal analysis is based on the theory of hyperfunctions ([9], [10], [15]). Here, we give only a rough sketch. A more complete treatment of microlocal filtering can be found in R. Ashino, C. Heil, M. Nagase, and R. Vaillancourt [2] (See also [3]). The important point is to find directions in which a hyperfunction can be continued analytically.

Let $\Omega \subset \mathbb{R}^n$ be an open set, and $\Gamma \subset \mathbb{R}^n$ be a convex open cone with vertex at 0. From now on, every cone is assumed to have vertex at 0. The set $\Omega + i\Gamma \subset \mathbb{C}^n$ is called a *wedge*. An *infinitesimal wedge* $\Omega + i\Gamma 0$ is an open set $U \subset \Omega + i\Gamma$ which approaches asymptotically to Γ as the imaginary part tends to 0. (Figure 1.)

A *hyperfunction* $f(x)$ can be defined as a sum

$$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0), \qquad x \in \Omega, \tag{2.2}$$

of formal boundary values

$$F_j(x + i\Gamma_j 0) = \lim_{y\to 0;\ x+iy\in\Omega+i\Gamma_j 0} F_j(x+iy) \tag{2.3}$$

of holomorphic functions $F_j(z)$ in infinitesimal wedges $\Omega + i\Gamma_j 0$.

A hyperfunction is said to be *micro-analytic* in the direction $\xi_0 \in \mathbb{S}^{n-1}$ at $x_0 \in \mathbb{R}^n$, or in short, at (x_0, ξ_0), if there exists a neighborhood Ω of x_0 and holomorphic functions F_j on infinitesimal wedges $\Omega + i\Gamma_j 0$ such that $f = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0)$ and

$$\Gamma_j \cap \{\, y \in \mathbb{R}^n \ : \ y\cdot\xi_0 < 0 \,\} \neq \emptyset \tag{2.4}$$

for all j.

A simple aspect of the relation between micro-analyticity and Fourier transform is given as follows.

Lemma 2.2. *Let $\Gamma \subset \mathbb{R}^n$ be a closed cone and $x_0 \in \mathbb{R}^n$. For $f \in \mathscr{S}'(\mathbb{R}^n)$, if there exists $g \in \mathscr{S}'(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{g} \subset \Gamma$ and $f - g$ is analytic in a neighborhood of x_0, then f is micro-analytic at (x_0, ξ) for every $\xi \in \Gamma^c \cap \mathbb{S}^{n-1}$, where Γ^c denotes the complement of Γ.*

2.3. **1-D Orthonormal Wavelets for Microlocal Filtering.** Define $\psi_\pm$ by $\widehat{\psi}_\pm = \chi_{[\pm 2\pi, \pm 4\pi]}$ (Figure 2). Then, $\{\psi_+, \psi_-\}$ is a system of orthonor-

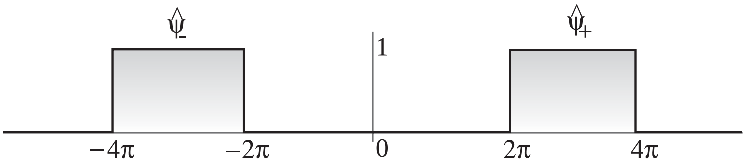

FIGURE 2. The Fourier transform of $\psi_\pm$.

mal wavelet functions. Define the orthogonal projections $\mathcal{P}_\pm$ by

$$\mathcal{P}_\pm f := \sum_{j,k\in\mathbb{Z}} \langle f, (\psi_\pm)_{jk}\rangle \, (\psi_\pm)_{jk} \ . \tag{2.5}$$

Then $f = \mathcal{P}_+ f + \mathcal{P}_- f$ and $\operatorname{supp}(\mathcal{P}_\pm f)^\wedge \subset \overline{\mathbb{R}_\pm}$, respectively. Hence, $\mathcal{P}_+ f(x)$ (resp. $\mathcal{P}_- f(x)$) is a boundary value of a holomorphic function on $\{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$ (resp. $\{ z \in \mathbb{C} : \operatorname{Im} z < 0 \}$). Thus, f is decomposed into two parts with micro-analytic direction ∓ 1. Each of $\mathcal{P}_\pm f$ can be decomposed and reconstructed by the usual filtering processes using wavelets. Since each wavelet function has a scaling function, there are two scaling functions. Hence those wavelets are called *multiwavelets* in [2].

2.4. n**-D Orthonormal Wavelets for Microlocal Filtering.** In the n-dimensional case, the set of all micro-analytic directions is the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$, which is an infinite set for $n \geq 2$. It is possible for the orthonormal wavelet basis constructed in [2] to tell fairly well in which directions f is micro-analytic. The price to pay to get good angular resolution in $\mathbb{S}^{n-1}$ is the need for many wavelet functions.

Definition 2.3. For a closed cube $Q \subset \mathbb{R}^n$, define ψ_Q by

$$\widehat{\psi_Q}(\xi) := \chi_{2\pi Q}(\xi) \ ,$$

where $\chi_{2\pi Q}$ is the *characteristic function* of the cube $2\pi Q$ defined by

$$\chi_{2\pi Q}(\xi) := \begin{cases} 1, & \xi \in 2\pi Q \ , \\ 0, & \text{otherwise} \ . \end{cases}$$

For an interval $I \subset \mathbb{R}$, we can easily compute that

$$\chi_I^\vee(t) = \frac{|I|}{2\pi} \exp(ic_I t) \operatorname{sinc}\Big(\frac{|I|}{2\pi} t\Big) \ , \tag{2.6}$$

where χ_I is the characteristic function of I, $|I|$ is the length of I, c_I is the center of I, and

$$\operatorname{sinc} t := \frac{\sin(\pi t)}{\pi t} \quad \text{for} \ \ t \neq 0, \qquad \operatorname{sinc} 0 := 1 \ . \tag{2.7}$$

From this, we can easily see that if $Q = \prod_{\nu=1}^n [c_\nu - \sigma_\nu/2, c_\nu + \sigma_\nu/2]$, where $c_\nu \in \mathbb{R}$ and $\sigma_\nu > 0$, then

$$\psi_Q(x) = \prod_{\nu=1}^n \{\sigma_\nu \exp(i2\pi c_\nu x_\nu) \operatorname{sinc}(\sigma_\nu x_\nu)\} \ . \tag{2.8}$$

Thus, $\psi_Q \notin L_1(\mathbb{R}^n)$ and $\psi_Q \in \bigcap_{1<r\leq\infty} L_r(\mathbb{R}^n)$. We shall see later that ψ_Q belongs to better spaces.

Definition 2.4. (1) By convention, we use $[a,b] := \{\, x \in \mathbb{R} \ : \ \min\{a,b\} \leq x \leq \max\{a,b\} \,\}$ even when $a \geq b$.
(2) For $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in \mathbb{R}^n$, the element-wise product is denoted by $a.{*}b := (a_1 b_1, \dots, a_n b_n) \in \mathbb{R}^n$ and the element-wise quotient is denoted by $a./b := (a_1/b_1, \dots, a_n/b_n)$. (Matlab convention)
(3) For $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in H := \{\pm 1\}^n = \{\pm\}^n$, set

$$\Gamma_\eta := \{\, \omega \in \mathbb{R}^n \ : \ \eta_\nu \omega_\nu > 0, \ \nu = 1, \dots, n \,\},$$

which is an open orthant in $\mathbb{R}^n$, and set

$$Q_\eta := \prod_{\nu=1}^n [0, \eta_\nu] = \{\, \omega \in \mathbb{R}^n \ : \ 0 \leq \eta_\nu \omega_\nu \leq 1, \ \nu = 1, \dots, n \,\} \ ,$$

which is a unit cube in the closed orthant $\overline{\Gamma_\eta}$. (See Figure 3 for $n = 2$.)
(4) For $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in E := \{\, 0, 1 \,\}^n \setminus \{\, (0, \dots, 0) \,\}$ and $\eta \in H$, consider the $2^n \times (2^n - 1)$ unit cubes

$$\begin{aligned} Q_\eta + \varepsilon.{*}\eta &= \prod_{\nu=1}^n [\eta_\nu \varepsilon_\nu, \eta_\nu(\varepsilon_\nu + 1)] \\ &= \{\, \omega \in \mathbb{R}^n \ : \ \varepsilon_\nu \leq \eta_\nu \omega_\nu \leq \varepsilon_\nu + 1, \ \nu = 1, \dots, n \,\} \ . \end{aligned} \tag{2.9}$$

(See Figure 4 for $n = 2$.) For $(\varepsilon, \eta) \in E \times H$, and $\rho = \rho(\varepsilon, \eta) \in \mathbb{Z}_+$, let $\mathscr{Q}_{\rho,\varepsilon,\eta}$

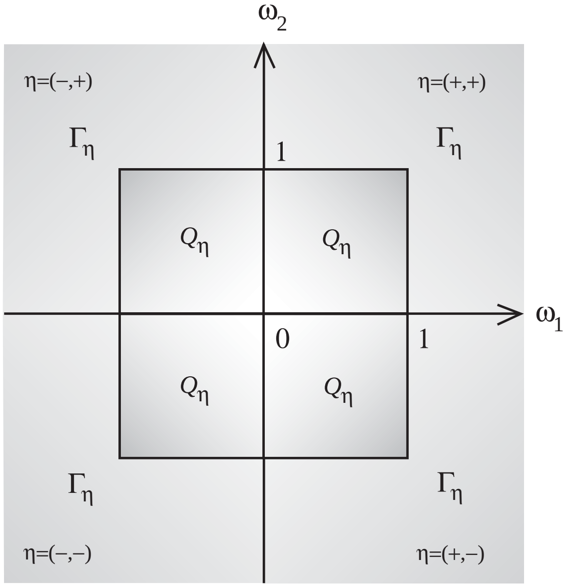

FIGURE 3. Orthants (Quadrants) Γ_η and Cubes (Squares) Q_η $(n = 2)$

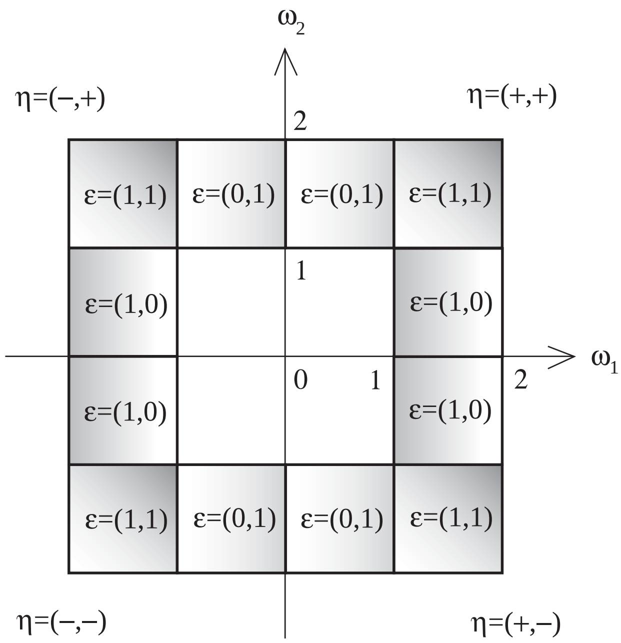

FIGURE 4. Cubes (Squares) $Q_\eta + \varepsilon{.*}\eta$ $(n = 2)$

be the collection of $2^{\rho n}$ unit cubes that cover $2^\rho(Q_\eta + \varepsilon{.*}\eta)$ with overlaps of measure zero, i.e.,

$$\mathscr{Q}_{\rho,\varepsilon,\eta} := \left\{ \prod_{\nu=1}^{n} [\eta_\nu l_\nu, \eta_\nu(l_\nu + 1)] + 2^\rho(\varepsilon{.*}\eta) \ :\right.$$

$$
\left. 0 \le l_1, \dots, l_n \le 2^\rho - 1,\ l_1, \dots, l_n \in \mathbb{Z} \right\}
$$
$$
= \left\{ Q_\eta + (2^\rho \varepsilon + l).{*}\eta \ : \ 0 \le l_1, \dots, l_n \le 2^\rho - 1,\ l_1, \dots, l_n \in \mathbb{Z} \right\} .
$$

(See Figure 5.)

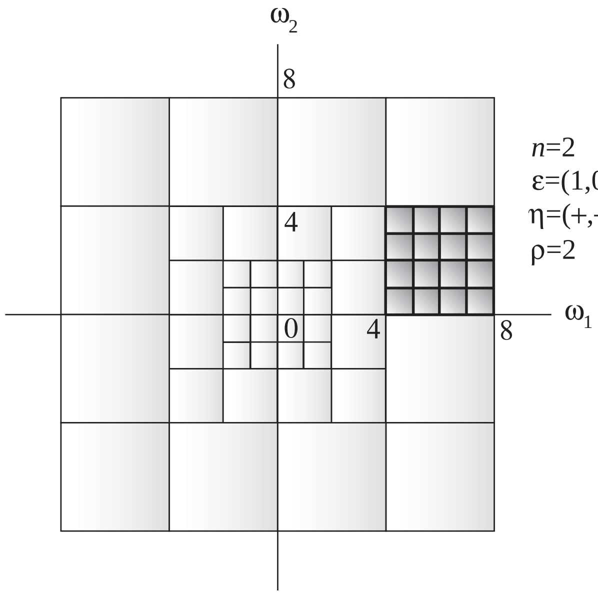

FIGURE 5. $\mathscr{Q}_{\rho,\varepsilon,\eta}$ $(n = 2,\ \varepsilon = (1,0),\ \eta = (+,+),\ \rho = 2)$: $2^{\rho n}$ unit cubes covering $2^\rho(Q_\eta + \varepsilon.{*}\eta)$.

From now on, we fix an arbitrary $\rho = \rho(\varepsilon, \eta) \in \mathbb{Z}_+$ for each $(\varepsilon, \eta) \in E \times H$, and set $\mathscr{Q}_\rho := \bigcup_{(\varepsilon,\eta)\in E\times H} \mathscr{Q}_{\rho(\varepsilon,\eta),\varepsilon,\eta}$. (In the definition of $\mathscr{Q}_\rho$ in [2], 2π should be eliminated, and some argument should be trivially modified afterward.) For $Q \in \mathscr{Q}_\rho$, there exists a unique $\eta \in H$ such that $Q \subset \overline{\Gamma_\eta}$. This η is denoted by $\eta(Q)$.

The most basic fact about the family $\{ \psi_Q \}_{Q \in \mathscr{Q}_\rho}$ is that it is a system of orthonormal wavelet functions as follows.

Theorem 2.5. ([2]) $\{ (\psi_Q)_{j,k} \ : \ Q \in \mathscr{Q}_\rho,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n \}$ *is an orthonormal basis for* $L_2(\mathbb{R}^n)$. *That is,*

$$
\langle (\psi_Q)_{j,k}, (\psi_{Q'})_{j',k'} \rangle = \begin{cases} 0 & \text{if } (Q,j,k) \ne (Q',j',k') \\ 1 & \text{if } (Q,j,k) = (Q',j',k') \end{cases} , \tag{2.10}
$$

and every $f \in L_2(\mathbb{R}^n)$ *is uniquely expressed in the form*

$$
f(x) = \sum_{Q \in \mathscr{Q}_\rho} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \langle f, (\psi_Q)_{j,k} \rangle \ (\psi_Q)_{j,k}(x) . \tag{2.11}
$$

By the orthogonality, the convergence is *unconditional*, that is, it converges for every choice of the order of summation. For unconditional convergence, see, for example, [12], [5], [21], etc. For each $Q \in \mathscr{Q}_\rho$, the orthogonal projection $\mathcal{P}_Q f := \sum_{j,k} \langle f, (\psi_Q)_{j,k} \rangle (\psi_Q)_{j,k}$ satisfies $\operatorname{supp} \widehat{\mathcal{P}_Q f} \subset \Gamma_Q$, where Γ_Q is the closed cone generated by $\bigcup_{j\in\mathbb{Z}} 2^j Q$, that is, $\Gamma_Q := \{ k\eta \ : \ \eta \in \bigcup_{j\in\mathbb{Z}} 2^j Q, \ k \in \overline{\mathbb{R}_+} \}$. Hence, $\mathcal{P}_Q f$ is micro-analytic in the direction $\xi \in {\Gamma_Q}^c \cap \mathcal{S}^{n-1}$ at every $x \in \mathbb{R}^n$. The larger ρ we take, the smaller Γ_Q we have.

In this article, we consider the expansion (2.11) in L_p spaces and other related spaces. The expansion in $L_p(\mathbb{R}^n)$ does not follow from the general theory (see, for example, [5],[21]), since ψ_Q is not well localized. The main result of this article is the following.

Theorem 2.6. *Assume* $1 < p < \infty$ *and* $f \in L_p(\mathbb{R}^n)$.
(1) *For each* $Q \in \mathscr{Q}_\rho$ *and each* $j \in \mathbb{Z}$, *the series*

$$\sum_{k\in\mathbb{Z}^n} \langle f, (\psi_Q)_{j,k} \rangle \ (\psi_Q)_{j,k}$$

converges unconditionally in $L_p(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$, *that is, it converges both in the* L_p*-norm and uniformly on* $\mathbb{R}^n$. *The sum is denoted by* $\mathcal{P}_j^Q f$.
(2) *The series* $\sum_{Q\in\mathscr{Q}_\rho, j\in\mathbb{Z}} \mathcal{P}_j^Q f$ *converges unconditionally to* f *in* $L_p(\mathbb{R}^n)$.

Thus, the wavelet expansion (2.11) is valid also in $L_p(\mathbb{R}^n)$. Note that $L_p(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n) = \bigcap_{p\le r\le\infty} L_r(\mathbb{R}^n)$ by the interpolation theorem. In this article, this type of convergence is called *stepwise* unconditional convergence. The total unconditionality of the convergence of $\sum_{Q,j,k} \langle f, (\psi_Q)_{j,k} \rangle \ (\psi_Q)_{j,k}$ is an open problem for $p \neq 2$. We shall also give similar results for some other spaces.

The result (1) turns out to be a variant of the L_p version of the sampling theorem, which has been given by F. Gensun [4]. We shall give a simpler proof than his.

3. Function Spaces

As we already said, we consider L_p spaces. We also consider $\widehat{L_p}(\mathbb{R}^n) = \{ \widehat{f} \ : \ f \in L_p(\mathbb{R}^n) \}$ and Sobolev type spaces.

Definition 3.1. Set $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ and set $\langle D \rangle^s f := \{ \langle \xi \rangle^s \widehat{f}(\xi) \}^\vee$ for $s \in \mathbb{R}$ and $f \in \mathscr{S}'(\mathbb{R}^n)$. Let $1 \le p \le \infty$. Define the Sobolev type spaces as follows.

$$L_p^s(\mathbb{R}^n) := \{ f \in \mathscr{S}'(\mathbb{R}^n) \ : \ \langle D \rangle^s f \in L_p(\mathbb{R}^n) \} \ , \tag{3.1}$$

$$L_p^{\infty}(\mathbb{R}^n) := \bigcap_{s\in\mathbb{R}} L_p^s(\mathbb{R}^n) \ , \qquad L_p^{-\infty}(\mathbb{R}^n) := \bigcup_{s\in\mathbb{R}} L_p^s(\mathbb{R}^n) \ , \tag{3.2}$$

$$M_p^s(\mathbb{R}^n) := \{ f \in \mathscr{S}'(\mathbb{R}^n) \ : \ \langle D\rangle^s f \in \widehat{L_p}(\mathbb{R}^n) \} \ , \tag{3.3}$$

$$M_p^{\infty}(\mathbb{R}^n) := \bigcap_{s\in\mathbb{R}} M_p^s(\mathbb{R}^n) \ , \qquad M_p^{-\infty}(\mathbb{R}^n) := \bigcup_{s\in\mathbb{R}} M_p^s(\mathbb{R}^n) \ . \tag{3.4}$$

As is well-known, if $s \in \mathbb{Z}_+$, then

$$L_p^s(\mathbb{R}^n) = \{ f \in \mathscr{S}'(\mathbb{R}^n) \ : \ \partial_x^\alpha f \in L_p(\mathbb{R}^n) \text{ for } \alpha \in \mathbb{Z}_+^n, |\alpha| \le s \} \ , \tag{3.5}$$

$$M_p^s(\mathbb{R}^n) = \{ f \in \mathscr{S}'(\mathbb{R}^n) \ : \ \partial_x^\alpha f \in \widehat{L_p}(\mathbb{R}^n) \text{ for } \alpha \in \mathbb{Z}_+^n, |\alpha| \le s \} \ . \tag{3.6}$$

We also use some spaces of band-limited functions (distributions).

Definition 3.2. For a bounded closed set $Q \subset \mathbb{R}^n$, set

$$E^Q := \{ f \in \mathscr{S}'(\mathbb{R}^n) \ : \ \operatorname{supp} \widehat{f} \subset 2\pi Q \} \ .$$

Further, for $1 \le p \le \infty$, set

$$B_p^Q := E^Q \cap L_p(\mathbb{R}^n) \ .$$

The space B_p^Q is a closed subspace of $L_p(\mathbb{R}^n)$ and we consider the induced topology from $L_p(\mathbb{R}^n)$. Similarly, for $1 \le p \le \infty$, set

$$PW_p^Q := E^Q \cap \widehat{L_p}(\mathbb{R}^n) \ .$$

The space PW_p^Q is a closed subspace of $\widehat{L_p}(\mathbb{R}^n)$ and we consider the induced topology from $\widehat{L_p}(\mathbb{R}^n)$. We followed the notation of [6], where spaces like B_p^Q are called *Bernstein spaces*, and those like PW_p^Q are called *Paley-Wiener spaces*. In some other references, spaces like B_p^Q are called Paley-Wiener spaces without considering PW_p^Q.

Remark 3.3. (1) An element of E^Q is an entire function of exponential type. Especially, $B_p^Q \hookrightarrow \mathscr{O}(\mathbb{C}^n)$ and $PW_p^Q \hookrightarrow \mathscr{O}(\mathbb{C}^n)$, where $\mathscr{O}(\mathbb{C}^n)$ is the space of entire functions with the topology of uniform convergence on compact sets, and $X \hookrightarrow Y$ denotes that X is continuously embedded in Y.
(2) We have $B_p^Q \hookrightarrow B_r^Q$ if $1 \le p \le r \le \infty$. We also have $B_p^Q \hookrightarrow L_r^s(\mathbb{R}^n)$ for every $r \in [p,\infty]$ and $s \in \mathbb{R} \cup \{\infty\}$. Further, when $r = p$, the induced topologies from $L_p^s(\mathbb{R}^n)$ to B_p^Q coincide for all $s \in \mathbb{R}$.

Similarly, we have $PW_p^Q \hookrightarrow PW_r^Q$ if $1 \le r \le p \le \infty$. We also have $PW_p^Q \hookrightarrow M_r^s(\mathbb{R}^n)$ for every $r \in [1,p]$ and $s \in \mathbb{R} \cup \{\infty\}$. Further, when $r = p$, the induced topologies from $M_p^s(\mathbb{R}^n)$ to PW_p^Q coincide for all $s \in \mathbb{R}$.

Note that $\partial_x^\alpha \psi_Q \notin L_1(\mathbb{R}^n)$ for every $\alpha \in \mathbb{Z}_+^n$, and that

$$\psi_Q \in \left(\bigcap_{1<p\le\infty} B_p^Q \right) \cap \left(\bigcap_{1\le p\le\infty} PW_p^Q \right) \tag{3.7}$$

$$\subset \left(\bigcap_{1<p\le\infty} L_p^\infty(\mathbb{R}^n) \right) \cap \left(\bigcap_{1\le p\le\infty} M_p^\infty(\mathbb{R}^n) \right) , \tag{3.8}$$

for every closed cube $Q \subset \mathbb{R}^n$. By (3.8), the inner product $\langle f, (\psi_Q)_{j,k} \rangle = \frac{1}{(2\pi)^n} \left\langle \widehat{f}, \widehat{(\psi_Q)_{j,k}} \right\rangle$ is well-defined, for every $Q \in \mathscr{Q}_\rho$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and every $f \in \left(\bigcup_{1\le p<\infty} L_p^{-\infty}(\mathbb{R}^n) \right) \cup \left(\bigcup_{1\le p\le\infty} M_p^{-\infty}(\mathbb{R}^n) \right)$.

4. Expansions

The following two theorems give the validity of the wavelet expansion (2.11) in the spaces $L_p^s(\mathbb{R}^n)$ and $M_p^s(\mathbb{R}^n)$ for $1 < p < \infty$. Theorem 2.6 is a special case $s = 0$ of Theorem 4.1.

Theorem 4.1. *Suppose that $1 < p < \infty$, $s \in \mathbb{R}$, and $f \in L_p^s(\mathbb{R}^n)$. Then we have:*

(1) *For every $Q \in \mathscr{Q}_\rho$ and every $j \in \mathbb{Z}$, the series*

$$\sum_{k\in\mathbb{Z}^n} \langle f, (\psi_Q)_{j,k} \rangle \, (\psi_Q)_{j,k}$$

converges unconditionally in $B_p^{2^jQ} \left(\hookrightarrow L_p^\infty(\mathbb{R}^n) \cap L_\infty^\infty(\mathbb{R}^n) \right)$. The sum is denoted by $\mathcal{P}_j^Q f$.

(2) $\sum_{Q\in\mathscr{Q}_\rho} \sum_{j\in\mathbb{Z}} \mathcal{P}_j^Q f$ *converges unconditionally to f in $L_p^s(\mathbb{R}^n)$.*

Theorem 4.2. *Suppose that $1 < p < \infty$, $s \in \mathbb{R}$, and $f \in M_p^s(\mathbb{R}^n)$. Then we have:*

(1) *For every $Q \in \mathscr{Q}_\rho$ and every $j \in \mathbb{Z}$, the limit of rectangular sum*

$$\lim_{M\to\infty} \sum_{k\in\mathbb{Z}^n;\ |k_\nu|\le M} \langle f, (\psi_Q)_{j,k} \rangle \, (\psi_Q)_{j,k}$$

exists in $PW_p^{2^jQ} \left(\hookrightarrow M_p^\infty(\mathbb{R}^n) \cap M_1^\infty(\mathbb{R}^n) \right)$. The limit is also denoted by $\mathcal{P}_j^Q f$.

(2) $\sum_{Q\in\mathscr{Q}_\rho} \sum_{j\in\mathbb{Z}} \mathcal{P}_j^Q f$ *converges unconditionally to f in $M_p^s(\mathbb{R}^n)$.*

Remark 4.3. (1) The unconditionality of the convergence in Theorem 4.1 is stepwise. Even if $s = 0$, the total unconditionality of the convergence

$$\sum_{Q\in\mathscr{Q}_\rho} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \langle f, (\psi_Q)_{j,k} \rangle \, (\psi_Q)_{j,k}$$

in $L_p(\mathbb{R}^n)$ is an open problem unless $p = 2$.
(2) If $p \neq 2$, then the convergence to $\mathcal{P}_j^Q f$ in Theorem 4.2 (1) is not unconditional in general even if $s = 0$. In fact, we shall see that the series is just the Fourier series expansion in $L_p(2\pi 2^j Q)$ (see Section 8), and $\{e^{ik\cdot\xi}\}_{k\in\mathbb{Z}^n}$ is not an unconditional basis of $L_p([-\pi,\pi]^n)$ ([18], Chapter II, Example 14.3; [20], II.D.9).

5. Preliminaries

Let Q be a closed cube in $\mathbb{R}^n$. We define the "*partial sum operator* ([19])" S_Q as follows, not only as an operator $L_2(\mathbb{R}^n) \to B_2^Q \subset L_2(\mathbb{R}^n)$ but also as an operator $L_p^{-\infty}(\mathbb{R}^n) \to B_p^Q$ for $1 < p < \infty$ and $M_p^{-\infty}(\mathbb{R}^n) \to PW_p^Q$ for $1 \leq p \leq \infty$.

Definition 5.1. (1) For $f \in L_2(\mathbb{R}^n)$ or more generally for $f \in M_p^{-\infty}(\mathbb{R}^n)$ ($1 \leq p \leq \infty$), define S_Q by $\widehat{(S_Q f)}(\xi) := \chi_{2\pi Q}(\xi)\widehat{f}(\xi)$. Note that we use $\chi_{2\pi Q}$, while χ_Q is used in [19].
(2) Let $1 < p < \infty$. For $f \in L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$, by Theorem 4 in Chapter IV of [19], we have $S_Q f \in L_p(\mathbb{R}^n)$ and there exists a positive constant A_p independent of n, Q and f such that

$$\|S_Q f\|_{L_p(\mathbb{R}^n)} \leq A_p^{\,n} \|f\|_{L_p(\mathbb{R}^n)} \quad . \tag{5.1}$$

Thus, S_Q is uniquely extended to a bounded operator from $L_p(\mathbb{R}^n)$ to B_p^Q, and hence $S_Q f$ is defined for $f \in L_p(\mathbb{R}^n)$, $1 < p < \infty$.
(3) Let $1 < p < \infty$ and $s \in \mathbb{R}$. We take $\phi \in \mathscr{S}(\mathbb{R}^n)$ such that $\phi(\xi) = 1$ in a neighborhood of $2\pi Q$. If $f \in L_p^s(\mathbb{R}^n)$, then $\big(\phi\,\widehat{f}\,\big)^\vee \in L_p(\mathbb{R}^n)$, since $(\phi\,\langle\xi\rangle^{-s})^\vee \in \mathscr{S}(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n)$ and $\big(\phi\,\widehat{f}\,\big)^\vee = (\phi\,\langle\xi\rangle^{-s})^\vee * (\langle D\rangle^s f)$. Hence, we can define $S_Q f$ for $f \in L_p^s(\mathbb{R}^n)$ by $S_Q f := S_Q(\big(\phi\,\widehat{f}\,\big)^\vee) \in B_p^Q$. It is easy to see that this definition is independent of the choice of ϕ.

We have the following basic properties of S_Q on $L_p^s(\mathbb{R}^n)$.

Proposition 5.2. *Let $1 < p < \infty$ and $s \in \mathbb{R}$.*
(1) *Let Σ be a union of finite number of hyperplanes in $\mathbb{R}^n$. If $g \in L_p^s(\mathbb{R}^n)$ satisfies* $\operatorname{supp}\widehat{g} \subset \Sigma$, *then $g = 0$.*

Hence, $g := S_Q f$, defined in Definition 5.1 (3) *for $f \in L_p^s(\mathbb{R}^n)$, is the unique element of $L_p^{-\infty}(\mathbb{R}^n)$ satisfying*

$$\widehat{g}|_{2\pi\mathring{Q}} = \widehat{f}|_{2\pi\mathring{Q}} \quad \text{in} \quad \mathscr{D}'(2\pi\mathring{Q}) \ , \tag{5.2}$$

$$\widehat{g}|_{2\pi Q^c} = 0 \qquad \text{in} \quad \mathscr{D}'(2\pi Q^c) \ , \tag{5.3}$$

where $\mathring{Q}$ *denotes the interior of* Q.
(2) *For* $f \in L_p^s(\mathbb{R}^n)$, *we have* $S_Q f = \psi_Q * f$. *Note that the right hand side can be defined in* $\bigcap_{p<r\le\infty} L_r(\mathbb{R}^n)$ *by Young's inequality, since*

$$\psi_Q * f = (\chi_{2\pi Q}\langle\cdot\rangle^{-s})^{\vee} * (\langle D\rangle^s f) \ ,$$
$$(\chi_{2\pi Q}\ \langle\cdot\rangle^{-s})^{\vee} = (\chi_{2\pi Q}\ \phi\ \langle\cdot\rangle^{-s})^{\vee} = \psi_Q * (\phi\ \langle\cdot\rangle^{-s})^{\wedge} \in \bigcap_{1<r\le\infty} L_r(\mathbb{R}^n) \ ,$$

where ϕ *is the function used in Definition* 5.1.

Remark 5.3. If $1 \le p \le 2$ and $s = 0$, then $\widehat{f} \in L_q(\mathbb{R}^n) \subset L_{1,loc}(\mathbb{R}^n)$, where $1/p + 1/q = 1$, and hence (1) of Proposition 5.2 is trivial. If $p > 2$, then $\widehat{f}$ does not necessarily belong to $L_{1,loc}(\mathbb{R}^n)$ (see, for example, [16]), and hence it is not trivial.

It is also worth noting that if Q is not a cube but a ball, then the estimate (5.1) is not valid unless $p \neq 2$ ([11], Theorem 3.5.6).

Proof of Proposition 5.2. Note that $\langle D\rangle^s$ is an isomorphism from $L_p^s(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$, and if $\operatorname{supp}\widehat{g} \subset \Sigma$, then $\operatorname{supp}(\langle D\rangle^s g)^{\wedge} \subset \Sigma$. Hence, we may assume $s = 0$. Let $g \in L_p(\mathbb{R}^n)$. We prove (1) by induction on the number of hyperplanes. If Σ itself is a hyperplane, we may assume by a rotation that $\Sigma = \{\, \xi \in \mathbb{R}^n \ : \ \xi_1 = a \,\}$ for some $a \in \mathbb{R}$. We write $\xi = (\xi_1, \xi')$, $\xi' = (\xi_2, \dots \xi_n)$, and similarly for x'. Then, $\operatorname{supp}\widehat{g} \subset \Sigma$ means $\widehat{g}(\xi) = \sum_{l=0}^{d} h_l(\xi')\,\delta^{(l)}(\xi_1 - a)$ for some $d \in \mathbb{Z}_+$ and some $h_l \in \mathscr{S}'(\mathbb{R}^{n-1})$ $(0 \le l \le d)$. Hence $g(x) = \sum_{l=0}^{d} (h_l)^{\vee}(x')(-ix_1)^l \exp(iax_1)/(2\pi)$. From $g \in L_p(\mathbb{R}^n)$, we obtain $(h_l)^{\vee} = 0$ for every l.

Assume that (1) is valid if the number of hyperplanes is less than d, and consider the case where Σ is a union of d hyperplanes $\{\, P_j \ : \ j = 1, \dots, d \,\}$, where $d \ge 2$. Let $\xi_0 \in P_1 \setminus \bigcup_{j=2}^{d} P_j$. We can take $\phi \in \mathscr{S}(\mathbb{R}^n)$ such that $\phi(\xi) = 1$ in a neighborhood of ξ_0 and $\operatorname{supp}\phi \cap \left(\bigcup_{j=2}^{d} P_j\right) = \emptyset$. Since $\phi^{\vee} \in \mathscr{S}(\mathbb{R}^n) \subset L_1(\mathbb{R}^n)$, we have $\phi^{\vee} * g \in L_p(\mathbb{R}^n)$, and $\operatorname{supp}\widehat{\phi^{\vee} * g} = \operatorname{supp}(\phi\,\widehat{g}) \subset P_1$. Thus, by the result in the case $d = 1$, we have $\phi\ \widehat{g} = 0$. Since ξ_0 is an arbitrary point of $P_1 \setminus \bigcup_{j=2}^{d} P_j$, we have $\operatorname{supp}\widehat{g} \subset \bigcup_{j=2}^{d} P_j$. By the assumption of the induction, we have $g = 0$.

(2) Let $f \in L_p^s(\mathbb{R}^n)$ and take r as $p < r < \infty$. Then, not only $\psi_Q * f$ but also $S_Q f$ belongs to $L_r(\mathbb{R}^n)$, since $S_Q f \in B_p^Q \subset B_r^Q \subset L_r(\mathbb{R}^n)$. Since $\operatorname{supp}\left(\widehat{S_Q f} - \chi_{2\pi Q}\widehat{f}\right) \subset \partial Q$, we have $S_Q f - \psi_Q * f = 0$ using (1) for r instead of p. □

We shall also use the following estimates. (Though we can extend this for more general setting using the theory of vector-valued integrals, we avoid to use big machinery.)

Lemma 5.4. *Let $1 \le p \le \infty$. Let (X, μ) and (Y, ν) be σ-finite complete measure spaces. If $F(x, y)$ is a measurable function on $X \times Y$, $F(\cdot, y) \in L_p(X)$ for almost every fixed $y \in Y$, and $\|F(\cdot, y)\|_{L_p(X)}$ is an integrable function of y on Y, then $F(x, y)$ is integrable with respect to y on Y for almost every fixed $x \in X$, and we have*

$$\left\| \int_Y F(\cdot, y)\, d\nu(y) \right\|_{L_p(X)} \le \int_Y \|F(\cdot, y)\|_{L_p(X)}\, d\nu(y) \ . \tag{5.4}$$

Proof. Take $q \in [1, \infty]$ as $1/p + 1/q = 1$. If $g \in L_q(X)$, then we have, by Fubini's theorem,

$$\begin{aligned} &\int_{X\times Y} |F(x, y)|\, |g(x)|\, d\mu(x) d\nu(y) \\ &\qquad = \int_Y \Big(\int_X |F(x, y)|\, |g(x)|\, d\mu(x) \Big)\, d\nu(y) \\ &\qquad \le \int_Y \|F(\cdot, y)\|_{L_p(X)}\, \|g\|_{L_q(X)}\, d\nu(y) \\ &\qquad = \int_Y \|F(\cdot, y)\|_{L_p(X)}\, d\nu(y)\ \|g\|_{L_q(X)} < \infty \ . \end{aligned} \tag{5.5}$$

This implies that $F(x, y)g(x)$ is integrable on $X \times Y$, and especially, $F(x, y)$ is integrable with respect to y on Y for almost every $x \in X$.

Thus, we have

$$\begin{aligned} \left\| \int_Y F(\cdot, y)\, d\nu(y) \right\|_{L_p(X)} &\le \left\| \int_Y |F(\cdot, y)|\, d\nu(y) \right\|_{L_p(X)} \\ &= \sup_{g\in L_q(X); \|g\|_{L_q(X)}\le 1} \left| \int_X \Big(\int_Y |F(x, y)|\, d\nu(y) \Big)\, |g(x)|\, d\mu(x) \right| \\ &= \sup_{g\in L_q(X); \|g\|_{L_q(X)}\le 1} \int_{X\times Y} |F(x, y)|\, |g(x)|\, d\mu(x) d\nu(y) \\ &\le \int_Y \|F(\cdot, y)\|_{L_p(X)}\, d\nu(y) \ , \end{aligned}$$

by using (5.5) again. □

6. L_p Version of the Sampling Theorem

In this section, we prove an L_p version of the sampling theorem, which turns out to be the essence of Theorem 2.6 (1). The slightly unusual formulation is more convenient for our purpose. Let $Q = \prod_{\nu=1}^{n} [c_\nu - \sigma_\nu/2, c_\nu + \sigma_\nu/2]$, where $c_\nu \in \mathbb{R}$ and $\sigma_\nu > 0$. Let $|Q| = |\sigma|_\star := |\sigma_1| \dots |\sigma_n|$ denote the volume of Q. Set $l_p(\mathbb{Z}^n) := \{ (a_m)_{m\in\mathbb{Z}^n} \ : \ \|(a_m)\|_{l_p(\mathbb{Z}^n)} := \big(\sum_{m\in\mathbb{Z}^n} |a_m|^p\big)^{1/p} < \infty \}$.

Theorem 6.1. *Let $1 < p < \infty$ and $f \in L_p(\mathbb{R}^n)$. Then we have the following four statements.*

(1) $(S_Q f)(m./\sigma) = \langle f, \psi_Q(\cdot - m./\sigma)\rangle$ *for* $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$.

(2) $\big((S_Q f)(m./\sigma)\big)_{m\in\mathbb{Z}^n} \in l_p(\mathbb{Z}^n)$. *Further, there exists a positive constant C_p independent of n and Q such that*

$$\big\|\big((S_Q f)(m./\sigma)\big)_m\big\|_{l_p(\mathbb{Z}^n)} \le C_p{}^n |Q|^{1/p} \|f\|_{L_p(\mathbb{R}^n)} \tag{6.1}$$

for every $f \in L_p(\mathbb{R}^n)$.

(3) *If $(y_m)_{m\in\mathbb{Z}^n} \in l_p(\mathbb{Z}^n)$, then the series $\Psi[(y_m)] := \sum_{m\in\mathbb{Z}^n} y_m \psi_Q(\cdot - m./\sigma)$ converges unconditionally in B_p^Q. Further, we have*

$$\|\Psi[(y_m)]\|_{L_p(\mathbb{R}^n)} \le C_q{}^n |Q|^{1-1/p} \|(y_m)\|_{l_p(\mathbb{Z}^n)} \ , \tag{6.2}$$

where C_q is the constant given in (2) for the conjugate exponent q to p, that is, $1/p + 1/q = 1$.

(4) *We have*

$$S_Q f = \frac{1}{|Q|} \sum_{m\in\mathbb{Z}^n} (S_Q f)(m./\sigma)\, \psi_Q(\cdot - m./\sigma) \ . \tag{6.3}$$

Note that if $f \in L_p(\mathbb{R}^n)$ and $\operatorname{supp} \widehat{f} \subset 2\pi Q$, then $S_Q f = f$. Further, we have by (2.8)

$$\psi_Q(x) = |\sigma|_\star e^{i2\pi c\cdot x} \operatorname{sinc}(\sigma.\!*x) \ , \tag{6.4}$$

where $\operatorname{sinc} x := \prod_{\nu=1}^n \operatorname{sinc} x_\nu$ and $\sigma.\!*x := (\sigma_1 x_1, \dots, \sigma_n x_n)$. Hence, it is easy to get the usual form of the sampling theorem from this theorem.

Corollary 6.2. *Let $1 < p < \infty$ and $\sigma = (\sigma_1, \dots, \sigma_n)$, $\sigma_\nu > 0$, $\nu = 1, \dots, n$. Assume that $f \in L_p(\mathbb{R}^n)$ and that $\operatorname{supp} \widehat{f} \subset \prod_{\nu=1}^n [-\sigma_\nu \pi, \sigma_\nu \pi]$. Then we have the following four statements.*

(1) $f(m./\sigma) = |\sigma|_\star \displaystyle\int_{\mathbb{R}^n} f(x) \operatorname{sinc}(\sigma.\!*x - m)\, dx$ *for* $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$.

(2) $\big(f(m./\sigma)\big)_m \in l_p(\mathbb{Z}^n)$. *Further, there exists a positive constant C_p independent of n, σ and f such that*

$$\begin{aligned} \frac{1}{C_p{}^n} |\sigma|_\star^{-1/p} \Big\|\big(f(m./\sigma)\big)_m\Big\|_{l_p(\mathbb{Z}^n)} &\le \|f\|_{L_p(\mathbb{R}^n)} \\ &\le C_p{}^n |\sigma|_\star^{-1/p} \Big\|\big(f(m./\sigma)\big)_m\Big\|_{l_p(\mathbb{Z}^n)} \ . \end{aligned} \tag{6.5}$$

(3) *If $(y_m)_m \in l_p(\mathbb{Z}^n)$, then the series $\sum_{m\in\mathbb{Z}^n} y_m \operatorname{sinc}(\sigma.\!*x - m)$ converges*

unconditionally in $L_p(\mathbb{R}^n)$*,* $L_\infty(\mathbb{R}^n)$*, and* $\mathscr{O}(\mathbb{C}^n)$*.*
(4) *We have*

$$f(x) = \sum_{m\in\mathbb{Z}^n} f(m./\sigma)\,\mathrm{sinc}(\sigma.{*}x - m)\ . \tag{6.6}$$

Remark 6.3. F. Gensun ([4], Theorems 1 and 2) proves this corollary (for $n = 1$), without mentioning the unconditionality of the convergence. Our proof is more elementary than his, by the extensive use of S_Q. Although there are 'similar' results before his, to our knowledge, they do not show L_p convergence. For example, Theorem 2.2 in [22] shows the absolute and uniform convergence on compact sets of $\mathbb{R}$ for $f \in L_p(\mathbb{R})$, but says nothing about L_p convergence.

Proof of Theorem 6.1. (1) Since $S_Q f = \psi_Q * f$ (Proposition 5.2 (2)) and since $\psi_Q(-x) = \overline{\psi_Q(x)}$, we have

$$\begin{aligned}(S_Q f)(m./\sigma) &= (\psi_Q * f)(m./\sigma) = \int_{\mathbb{R}^n} \psi_Q(m./\sigma - y) f(y)\,dy \\ &= \int_{\mathbb{R}^n} f(y)\overline{\psi_Q(y - m./\sigma)}\,dy = \langle f, \psi_Q(\cdot - m./\sigma)\rangle\ .\end{aligned} \tag{6.7}$$

(2) Take $\phi_0 \in \mathscr{S}(\mathbb{R})$ as $\widehat{\phi_0}(\tau) = 1$ in a neighborhood of $[-\pi,\pi]$. Set $\phi(x) = \prod_{\nu=1}^n \phi_0(x_\nu) \in \mathscr{S}(\mathbb{R}^n)$, and define $\phi_Q \in \mathscr{S}(\mathbb{R}^n)$ by $\widehat{\phi_Q}(\xi) := \widehat{\phi}\big((\xi - 2\pi c)./\sigma\big)$. We have $\widehat{\phi_Q}(\xi) = 1$ in a neighborhood of $2\pi Q$, and $\phi_Q(x) = |\sigma|_\star \phi(\sigma.{*}x)\exp(i2\pi c\cdot x)$. Since $S_Q f = \phi_Q * (S_Q f)$, we have

$$(S_Q f)(m./\sigma) = \int_{\mathbb{R}^n} \phi_Q(y)(S_Q f)(m./\sigma - y)\,dy, \tag{6.8}$$

and hence, by using Lemma 5.4 for $X = \mathbb{Z}^n$ with unit mass at each point and $Y = \mathbb{R}^n$, we have

$$\begin{aligned}&\Big\|\big((S_Q f)(m./\sigma)\big)_m\Big\|_{l_p(\mathbb{Z}^n)} \\ &\qquad\le \int_{\mathbb{R}^n} |\phi_Q(y)|\,\Big\|\big((S_Q f)(m./\sigma - y)\big)_m\Big\|_{l_p(\mathbb{Z}^n)}\,dy\ .\end{aligned} \tag{6.9}$$

Set $F(y) := \Big\|\big((S_Q f)(m./\sigma - y)\big)_m\Big\|_{l_p(\mathbb{Z}^n)}$, and F is a $(1/\sigma_1, \ldots, 1/\sigma_n)$-periodic function. Further, for every $l \in \mathbb{Z}^n$, we have

$$\begin{aligned}\int_{([0,1]^n + l)./\sigma} F(y)^p\,dy &= \int_{[0,1]^n./\sigma} F(y)^p\,dy \\ &= \int_{[0,1]^n./\sigma} \sum_{m\in\mathbb{Z}^n} |(S_Q f)(m./\sigma - y)|^p\,dy\end{aligned}$$

$$\begin{aligned}(6.10)\qquad &= \sum_{m\in\mathbb{Z}^n}\int_{[0,1]^n./\sigma} |(S_Q f)(m./\sigma - y)|^p\,dy\\ &= \int_{\mathbb{R}^n} |(S_Q f)(y')|^p\,dy'\\ &= \|S_Q f\|^p_{L_p(\mathbb{R}^n)} \le A_p{}^{np}\|f\|^p_{L_p(\mathbb{R}^n)}\ ,\end{aligned}$$

where $[0,1]^n./\sigma := \prod_{\nu=1}^n [0,1/\sigma_\nu]$ and $([0,1]^n+l)./\sigma := \prod_{\nu=1}^n [l_\nu/\sigma_\nu,(l_\nu+1)/\sigma_\nu]$. Thus, taking q as $1/p+1/q=1$, we have

$$\begin{aligned}&\Big\|\big((S_Q f)(m./\sigma)\big)_m\Big\|_{l_p(\mathbb{Z}^n)} \le \int_{\mathbb{R}^n} |\phi_Q(y)|\,F(y)\,dy\\ &= \sum_{l\in\mathbb{Z}^n}\int_{([0,1]^n+l)./\sigma} |\phi_Q(y)|\,F(y)\,dy\\ (6.11)\qquad &\le \sum_{l\in\mathbb{Z}^n}\Big(\int_{([0,1]^n+l)./\sigma} |\phi_Q(y)|^q\,dy\Big)^{1/q}\Big(\int_{([0,1]^n+l)./\sigma} F(y)^p\,dy\Big)^{1/p}\\ &\le \sum_{l\in\mathbb{Z}^n}\Big(\int_{([0,1]^n+l)./\sigma} |\phi_Q(y)|^q\,dy\Big)^{1/q} A_p{}^n\,\|f\|_{L_p(\mathbb{R}^n)}\ .\end{aligned}$$

Further, we have

$$\begin{aligned}\int_{([0,1]^n+l)./\sigma} |\phi_Q(y)|^q\,dy &= \int_{[0,1]^n+l} |\phi_Q(y'./\sigma)|^q\,dy'\frac{1}{|\sigma|_\star}\\ (6.12)\qquad &= \int_{[0,1]^n+l} |\phi(y')|^q\,dy'\frac{|\sigma|_\star^q}{|\sigma|_\star}\\ &= |\sigma|_\star^{q-1}\prod_{\nu=1}^n\Big(\int_{[0,1]+l_\nu} |\phi_0(y'_\nu)|^q\,dy'_\nu\Big)\ .\end{aligned}$$

By setting

$$\widetilde{C}_p := \sum_{l\in\mathbb{Z}}\Big(\int_{[0,1]+l} |\phi_0(t)|^q\,dt\Big)^{1/q} < \infty\ ,$$

we have

$$\sum_{l\in\mathbb{Z}^n}\Big(\int_{[0,1]^n+l} |\phi(y)|^q\,dy\Big)^{1/q} = \widetilde{C}_p{}^n\ ,$$

and hence

$$\Big\|\big((S_Q f)(m./\sigma)\big)_m\Big\|_{l_p(\mathbb{Z}^n)} \le A_p{}^n\widetilde{C}_p{}^n|Q|^{1/p}\|f\|_{L_p(\mathbb{R}^n)}\ .$$

Thus, we have the result with $C_p := A_p\widetilde{C}_p$.

(3) Let $(y_m)_m \in l_p(\mathbb{Z}^n)$ and J be an arbitrary finite subset of $\mathbb{Z}^n$. For every $g \in L_q(\mathbb{R}^n)$, we have

$$(6.13)\qquad \Big\langle \sum_{m\in J} y_m\psi_Q(\cdot - m./\sigma), g\Big\rangle = \sum_{m\in J} y_m\overline{\langle g, \psi_Q(\cdot - m./\sigma)\rangle}\ .$$

By putting $g_m := \langle g, \psi_Q(\cdot - m./\sigma)\rangle = (S_Q g)(m./\sigma)$, we have $(g_m)_m \in l_q(\mathbb{Z}^n)$ and $\|(g_m)_m\|_{l_q(\mathbb{Z}^n)} \le C_q{}^n |Q|^{1/q} \|g\|_{L_q(\mathbb{R}^n)}$ by using (2) for q instead of p. Hence, we have

$$
\begin{aligned}
\left|\left\langle \sum_{m\in J} y_m \psi_Q(\cdot - m./\sigma), g \right\rangle\right| &= \left|\sum_{m\in J} y_m \overline{g_m}\right| \\
&\le \|(y_m)_{m\in J}\|_{l_p(\mathbb{Z}^n)} \|(g_m)_m\|_{l_q(\mathbb{Z}^n)} \\
&\le \|(y_m)_{m\in J}\|_{l_p(\mathbb{Z}^n)} C_q{}^n |Q|^{1/q} \|g\|_{L_q(\mathbb{R}^n)}
\end{aligned}
\tag{6.14}
$$

for every $g \in L_q(\mathbb{R}^n)$. Thus, we have

$$
\left\|\sum_{m\in J} y_m \psi_Q(\cdot - m./\sigma)\right\|_{L_p(\mathbb{R}^n)} \le C_q{}^n |Q|^{1/q} \|(y_m)_{m\in J}\|_{l_p(\mathbb{Z}^n)} \ .
$$

This implies that $\sum_{m\in\mathbb{Z}^n} y_m \psi_Q(\cdot - m./\sigma)$ converges *unconditionally* in $L_p(\mathbb{R}^n)$, and hence in B_p^Q.

(4) Since $\{\, |Q|^{-1/2}\psi_Q(\cdot - m./\sigma)\,\}_{m\in\mathbb{Z}^n}$ is an orthonormal basis of B_2^Q, we have (6.3) for $f \in L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$. Since both sides of (6.3) are continuous in $f \in L_p(\mathbb{R}^n)$, we have (6.3) for every $f \in L_p(\mathbb{R}^n)$. □

7. Proof of Theorem 4.1

In this section, we prove Theorem 4.1.

First fix an arbitrary $Q \in \mathscr{Q}_\rho$ and $j \in \mathbb{Z}$. For (1), we may assume that $s = 0$. Since

$$
\widehat{(\psi_Q)_{j,k}}(\xi) = 2^{-nj/2} e^{-i2^{-j}k\cdot\xi} \widehat{\psi_Q}(2^{-j}\xi) = 2^{-nj/2} e^{-i2^{-j}k\cdot\xi} \chi_{2\pi 2^j Q}(\xi) \ , \tag{7.1}
$$

we have $(\psi_Q)_{j,k}(x) = 2^{-nj/2}\psi_{2^jQ}(x - 2^{-j}k)$. Hence, by Theorem 6.1 for $\sigma = 2^j(1, \dots, 1)$, if $f \in L_p(\mathbb{R}^n)$, then we have the following.

- $\langle f, (\psi_Q)_{j,k}\rangle = 2^{-nj/2}(S_{2^jQ} f)(2^{-j}k)$.
- $\big(\langle f, (\psi_Q)_{j,k}\rangle\big)_{k\in\mathbb{Z}^n} \in l_p(\mathbb{Z}^n)$ and

$$
\left\|\big(\langle f, (\psi_Q)_{j,k}\rangle\big)_k\right\|_{l_p(\mathbb{Z}^n)} \le C_p{}^n 2^{nj(1/p-1/2)} \|f\|_{L_p(\mathbb{R}^n)} \ .
$$

- $S_{2^jQ} f = \sum_{k\in\mathbb{Z}^n} \langle f, (\psi_Q)_{j,k}\rangle\, (\psi_Q)_{j,k}$, where convergence is unconditional in $B_p^{2^jQ}$.

Thus, we have (1) and $\mathcal{P}_j^Q f = S_{2^jQ} f$.

In order to prove (2), we prove the following lemma.

Lemma 7.1. *Let Q be a cube in $\mathbb{R}^n$ such that $|2Q \cap Q| = 0$. Then, for $1 < p < \infty$, there exists a constant $A_{Q,p}$ such that*

$$\left\| \sum_{j\in J} \epsilon_j S_{2^jQ} f \right\|_{L_p(\mathbb{R}^n)} \le A_{Q,p} \|f\|_{L_p(\mathbb{R}^n)} \tag{7.2}$$

for every finite subset $J \subset \mathbb{Z}$, every $f \in L_p(\mathbb{R}^n)$ and every choice of signs $\epsilon_j = \pm 1$ for each $j \in \mathbb{Z}$.

Proof. The proof is similar to that of (54) in [19], Chapter IV, §5. We refer to the equations and theorems in this book.

By an argument similar to the proof of (53) given in §5.3.2, which uses Rademacher functions and Theorems 3 and 4', there exists a constant $C_{Q,p}$ such that

$$\left\| \left(\sum_{j\in\mathbb{Z}} |S_{2^jQ} f|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^n)} \le C_{Q,p} \|f\|_{L_p(\mathbb{R}^n)} \tag{7.3}$$

for every $f \in L_p(\mathbb{R}^n)$. Though (53) is a one-dimensional result, we can proceed quite similarly for our n-dimensional case by considering $\{2^jQ\}_j$ instead of $\{I_m\}_m$. Note that we can take $\delta > 0$ such that the δ-neighborhood Q_δ of Q satisfies $0 \notin \overline{Q_\delta}$, and we can take $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(\xi) = 1$ in a neighborhood of $\overline{Q}$ and $\varphi(\xi) = 0$ for $\xi \notin Q_\delta$.

Let $f \in L_p(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$, $\operatorname{supp} \widehat{f} \subset \bigcup_{j\in\mathbb{Z}} 2^jQ$, and $g \in L_q(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$, where q is the conjugate exponent to p. Then,

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} f(x)\, \overline{g(x)}\, dx \right| &= \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi)\, \overline{\widehat{g}(\xi)}\, d\xi \right| \\
&= \left| \sum_{j\in\mathbb{Z}} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \chi_{2^jQ}(\xi) \widehat{f}(\xi)\, \overline{\chi_{2^jQ}(\xi)\widehat{g}(\xi)}\, d\xi \right| \\
&= \left| \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} (S_{2^jQ} f)(x)\, \overline{(S_{2^jQ} g)(x)}\, dx \right| \\
&\le \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |(S_{2^jQ} f)(x)| \cdot |(S_{2^jQ} g)(x)|\, dx \\
&\le \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} |(S_{2^jQ} f)(x)|^2 \right)^{1/2} \cdot \left(\sum_{j\in\mathbb{Z}} |(S_{2^jQ} g)(x)|^2 \right)^{1/2} dx \\
&\le \left\| \left(\sum_{j\in\mathbb{Z}} |S_{2^jQ} f|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^n)} \cdot \left\| \left(\sum_{j\in\mathbb{Z}} |S_{2^jQ} g|^2 \right)^{1/2} \right\|_{L_q(\mathbb{R}^n)} \\
&\le \left\| \left(\sum_{j\in\mathbb{Z}} |S_{2^jQ} f|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^n)} C_{Q,q} \|g\|_{L_q(\mathbb{R}^n)} \;,
\end{aligned}$$

where we used (7.3) replacing p by q. Hence, we have

$$\|f\|_{L_p(\mathbb{R}^n)} \le C_{Q,q} \left\| \left(\sum_{j\in\mathbb{Z}} |S_{2^jQ} f|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^n)} \tag{7.4}$$

for every $f \in L_p(\mathbb{R}^n)$ such that $\operatorname{supp} \widehat{f} \subset \bigcup_{j\in\mathbb{Z}} 2^j Q$.

Now, for an arbitrary $f \in L_p(\mathbb{R}^n)$, an arbitrary finite subset $J \subset \mathbb{Z}$, and arbitrary choices of signs $\epsilon_j = \pm 1$, put $g_J := \sum_{j\in J} \epsilon_j S_{2^jQ} f \in L_p(\mathbb{R}^n)$. Since $\operatorname{supp} \widehat{g_J} \subset \bigcup_{j\in\mathbb{Z}} 2^j Q$ and since $S_{2^jQ} g_J = \epsilon_j S_{2^jQ} f$ for $j \in J$ and $S_{2^jQ} g_J = 0$ for $j \notin J$, we have by applying (7.4) and (7.3)

$$\|g_J\|_{L_p(\mathbb{R}^n)} \le C_{Q,q} \left\| \left(\sum_{j\in J} |S_{2^jQ} f|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^n)} \le C_{Q,q} C_{Q,p} \|f\|_{L_p(\mathbb{R}^n)} \ . \tag{7.5}$$

Thus, we have (7.2) with $A_{Q,p} := C_{Q,q} C_{Q,p}$. □

Next, we show the following.

Lemma 7.2. *Set* $\mathscr{D}(\mathbb{R}^n) := \{ \phi \in C^\infty(\mathbb{R}^n) \ : \ \operatorname{supp} \phi \text{ is compact} \}$ *and* $\mathscr{D}_0 := \{ \phi \in \mathscr{D}(\mathbb{R}^n) \ : \ \operatorname{supp} \phi \subset \mathbb{R}^n \setminus \{0\} \}$. *Then,* $\widehat{\mathscr{D}_0}$ *is dense in* $L_p(\mathbb{R}^n)$.

Proof. Let T be an arbitrary element of $\big(L_p(\mathbb{R}^n)\big)' = L_q(\mathbb{R}^n)$, where q is the conjugate exponent to p, and assume that $\langle T, \widehat{\phi} \rangle = 0$ for every $\phi \in \mathscr{D}_0$. We have only to show that $T = 0$. The assumption means that $\langle \widehat{T}, \phi \rangle = 0$ for every $\phi \in \mathscr{D}_0$, and hence we have $\operatorname{supp} \widehat{T} \subset \{0\}$, that is, $\widehat{T} = \sum_{\alpha\in J} c_\alpha \delta^{(\alpha)}$ for some finite set $J \subset \mathbb{Z}_+^n$ and some $c_\alpha \in \mathbb{C}$. Thus, T is a polynomial. Since $T \in L_q(\mathbb{R}^n)$, we have $T = 0$. □

Fix $Q \in \mathscr{Q}_\rho$. For $f \in \widehat{\mathscr{D}_0}$, it is trivial that $\lim_{N\to\infty} \sum_{|j|\le N} \epsilon_j S_{2^jQ} f$ exists in $L_p(\mathbb{R}^n)$. Thus, by the well-known argument, which is given as a proposition below, we have that $\lim_{N\to\infty} \sum_{|j|\le N} \epsilon_j S_{2^jQ} f$ converges in $L_p(\mathbb{R}^n)$ for every $f \in L_p(\mathbb{R}^n)$ and every choice of signs $\epsilon_j = \pm 1$. This implies that $\sum_{j\in\mathbb{Z}} S_{2^jQ} f$ converges *unconditionally* in $L_p(\mathbb{R}^n)$ for every $f \in L_p(\mathbb{R}^n)$. (See, for example, [5], §5.2, Theorem 2.4.) We have $\sum_{Q\in\mathscr{Q}_\rho} \sum_{j\in\mathbb{Z}} S_{2^jQ} f = f$ trivially for $f \in L_p(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$, and hence for every $f \in L_p(\mathbb{R}^n)$ by continuity.

Proposition 7.3. *Let* T_j $(j \in \mathbb{Z}_+)$ *be a family of bounded linear operators from a Banach space* B_1 *to another Banach space* B_2. *Assume the following two conditions.*

(a) *There exists a constant* A *such that* $\| \sum_{j\in J} T_j f\|_{B_2} \le A \|f\|_{B_1}$ *for every* $f \in B_1$ *and every finite subset* $J \subset \mathbb{Z}_+$.

(b) *There exists a dense subset* D *of* B_1 *such that* $\sum_{j\in\mathbb{Z}_+} T_j f$ *converges for every* $f \in D$.

Then, $Sf := \sum_{j\in\mathbb{Z}_+} T_j f$ *converges for every* $f \in B_1$ *and* S *is a bounded linear operator from* B_1 *to* B_2*.*

Proof. Let $f \in B_1$. For every $\epsilon > 0$, there exists $g \in D$ such that $\|f-g\|_{B_1} < \epsilon$. By (b), there exists $N_0 \in \mathbb{Z}_+$ such that if $N \geq M \geq N_0$, then $\|\sum_{j=M}^N T_j g\|_{B_2} < \epsilon$. Hence, by (a), we have $\|\sum_{j=M}^N T_j f\|_{B_2} \leq \|\sum_{j=M}^N T_j g\|_{B_2} + \|\sum_{j=M}^N T_j(f-g)\|_{B_2} \leq \epsilon + A\|f-g\|_{B_1} \leq (1+A)\epsilon$. Thus, $Sf := \sum_{j\in\mathbb{Z}_+} T_j f$ converges for every $f \in B_1$. By (a), it is trivial that $\|Sf\|_{B_2} \leq A\|f\|_{B_1}$. □

Thus, we have proved (2) when $s = 0$. Since $L_p^s(\mathbb{R}^n) = \langle D\rangle^{-s}(L_p(\mathbb{R}^n))$ and since $\langle D\rangle^s(S_Q f) = S_Q(\langle D\rangle^s f)$ for $f \in L_p^s(\mathbb{R}^n)$, we have (2) for general $s \in \mathbb{R}$.

8. Proof of Theorem 4.2

In this section, we prove Theorem 4.2.

First fix an arbitrary $Q \in \mathscr{Q}_\rho$ and $j \in \mathbb{Z}$. Since we have (7.1) and

$$\begin{aligned}\langle f, (\psi_Q)_{j,k}\rangle &= \frac{1}{(2\pi)^n}\left\langle \widehat{f}, \widehat{(\psi_Q)_{j,k}}\right\rangle \\ &= \frac{1}{(2\pi)^n} 2^{-nj/2}\int_{\mathbb{R}^n} \widehat{f}(\xi)\chi_{2\pi 2^j Q}(\xi)\exp(i2^{-j}k\cdot\xi)\,d\xi \ ,\end{aligned}$$

we have

$$\begin{aligned}\langle f, (\psi_Q)_{j,k}\rangle\, \widehat{(\psi_Q)_{j,k}}(\xi) = \frac{1}{(2\pi 2^j)^n}\int_{2\pi 2^j Q} \widehat{f}(\xi)\exp(i2^{-j}k\cdot\xi)\,d\xi \\ \times \exp(-i2^{-j}k\cdot\xi)\chi_{2\pi 2^j Q}(\xi) \ .\end{aligned}$$

Thus, the Fourier transform of $\sum_{k\in\mathbb{Z}^n} \langle f, (\psi_Q)_{j,k}\rangle\, (\psi_Q)_{j,k}$ is the Fourier series expansion of $\chi_{2\pi 2^j Q}\widehat{f}$ on the cube $2\pi 2^j Q$. If $f \in M_p^s(\mathbb{R}^n)$, then $\chi_{2\pi 2^j Q}\widehat{f} \in L_p(2\pi 2^j Q)$, and hence $\sum_{k\in\mathbb{Z}^n;\ |k_\nu|\leq M} \langle f, (\psi_Q)_{j,k}\rangle\, \widehat{(\psi_Q)_{j,k}} \to \chi_{2\pi 2^j Q}\widehat{f}$ as $M \to \infty$ in $L_p(\mathbb{R}^n)$([1], Chapter 2, §1). Since $\langle\xi\rangle^r$ is bounded on $2\pi 2^j Q$ for every $r \in \mathbb{R}$, we have

$$\sum_{k\in\mathbb{Z}^n;\ |k_\nu|\leq M} \langle f, (\psi_Q)_{j,k}\rangle\, (\psi_Q)_{j,k} \to S_{2^j Q} f \text{ as } M \to \infty \text{ in } M_p^\infty(\mathbb{R}^n) \ .$$

Thus, we have (1) and $\mathcal{P}_j^Q f = S_{2^j Q} f$.

(2) is trivial by the definition of $M_p^s(\mathbb{R}^n)$.

9. Expression as a Sum of Boundary Values of Holomorphic Functions

We can use our wavelets as follows to express a function (distribution) in $L_p^{-\infty}(\mathbb{R}^n) \cup M_p^{-\infty}(\mathbb{R}^n)$ as a sum of boundary values of holomorphic functions.

As is already mentioned in Remark 3.3, if $f \in \mathscr{S}'(\mathbb{R}^n)$ is band-limited, then f can be extended as an entire function $f^\sharp$ on $\mathbb{C}^n$, and

$$\{f^\sharp(\cdot + iy)\}^\wedge(\xi) = \widehat{f}(\xi)e^{-y\cdot\xi} \tag{9.1}$$

for every $y \in \mathbb{R}^n$. In particular, $(\psi_Q)_{j,k}$ and $\mathcal{P}_j^Q f$ for $f \in L_p^{-\infty}(\mathbb{R}^n) \cup M_p^{-\infty}(\mathbb{R}^n)$ are extended to entire functions on $\mathbb{C}^n$.

Moreover, we have the following.

Proposition 9.1. (1) *If $f \in \mathscr{S}'(\mathbb{R}^n)$ and $\operatorname{supp} \widehat{f} \subset \overline{\Gamma_\eta}$, then there exists a holomorphic function $f^\sharp(z)$ on $\mathbb{R}^n + i\Gamma_\eta$ such that $f^\sharp(\cdot + iy) \in \mathscr{S}'(\mathbb{R}^n)$ and (9.1) is satisfied for $y \in \Gamma_\eta$. Further, $f^\sharp(\cdot + iy) \to f$ in $\mathscr{S}'(\mathbb{R}^n)$ as $y \in \Gamma_\eta$, $y \to 0$.*
(2) *Let $1 < p < \infty$ and $s \in \mathbb{R}$. If $f \in L_p^s(\mathbb{R}^n)$ (resp. $M_p^s(\mathbb{R}^n)$) and $\operatorname{supp} \widehat{f} \subset \overline{\Gamma_\eta}$, then*

(a) *$f^\sharp(\cdot + iy) \in L_p^\infty(\mathbb{R}^n)$ (resp. $M_p^\infty(\mathbb{R}^n)$) for $y \in \Gamma_\eta$,*
(b) *$f^\sharp(\cdot + iy) \to f$ in $L_p^s(\mathbb{R}^n)$ (resp. $M_p^s(\mathbb{R}^n)$) as $y \in \Gamma_\eta$, $y \to 0$.*

Proof. We take $\theta_0 \in \mathscr{S}(\mathbb{R})$ such that $\widehat{\theta_0}$ is real valued and $\widehat{\theta_0}(\tau) = e^{-\tau}$ in a neighborhood of $[0, \infty)$, and set $\theta(x) := \prod_{\nu=1}^n \theta_0(x_\nu) \in \mathscr{S}(\mathbb{R}^n)$. Further, for $y \in \mathbb{R}^n_\star$, set

$$\theta_y(x) := \frac{1}{|y|_\star}\theta(x./y) = \prod_{\nu=1}^n \Big\{ \frac{1}{|y_\nu|}\theta_0(x_\nu/y_\nu) \Big\} \in \mathscr{S}(\mathbb{R}^n) \ .$$

Note that $\widehat{\theta_y}(\xi) = \widehat{\theta}(y.*\xi)$, and hence, if $y \in \Gamma_\eta$, then $\widehat{\theta_y}(\xi) = e^{-y\cdot\xi}$ in a neighborhood of $\overline{\Gamma_\eta}$.

(1) Let $f \in \mathscr{S}'(\mathbb{R}^n)$ and $\operatorname{supp} \widehat{f} \subset \overline{\Gamma_\eta}$. If $y \in \Gamma_\eta$, then we have $e^{-y\cdot\xi}\widehat{f}(\xi) = \widehat{\theta_y}(\xi)\widehat{f}(\xi) \in \mathscr{S}'(\mathbb{R}^n)$. Hence, $f^\sharp(\cdot + iy) \in \mathscr{S}'(\mathbb{R}^n)$ can be defined by (9.1) for each $y \in \Gamma_\eta$, and it is also given by

$$f^\sharp(\cdot + iy) = \theta_y * f \quad , \quad y \in \Gamma_\eta \ . \tag{9.2}$$

Now, we can consider $f^\sharp(x + iy)$ as a distribution of (x, y) in $\mathbb{R}^n \times \Gamma_\eta$. Then, we have $\{\partial_x(f^\sharp(\cdot + iy))\}^\wedge(\xi) = i\xi\widehat{f}(\xi)e^{-y\cdot\xi}$ and $\{\partial_y(f^\sharp(\cdot + iy))\}^\wedge(\xi) = -\xi\widehat{f}(\xi)e^{-y\cdot\xi}$. Hence, $\partial_x(f^\sharp(x+iy)) = -i\partial_y(f^\sharp(x+iy))$. Thus, $f^\sharp(x+iy)$ satisfies the Cauchy-Riemann equation in $\mathbb{R}^n \times \Gamma_\eta$, and hence $f^\sharp$ is holomorphic in $\mathbb{R}^n + i\Gamma_\eta$.

Since $\widehat{\theta} \in \mathscr{S}(\mathbb{R}^n)$ and $\widehat{\theta}(0) = 1$, we can easily show that if $g \in \mathscr{S}(\mathbb{R}^n)$, then $\widehat{\theta}(y.{*}\xi)\,\widehat{g}(\xi) \to \widehat{g}(\xi)$ in $\mathscr{S}(\mathbb{R}^n)$ as $y \to 0$. Hence, for $f \in \mathscr{S}'(\mathbb{R}^n)$ and $g \in \mathscr{S}(\mathbb{R}^n)$, we have $\langle \theta_y * f, g\rangle = \langle f, \theta_y * g\rangle \to \langle f, g\rangle$ as $y \to 0$.

(2) First, we prove the case of $L_p^s(\mathbb{R}^n)$. For $y \in \Gamma_\eta$ and for every $r \in \mathbb{R}$, we have $\langle D\rangle^r\{f^\sharp(\cdot + iy)\} = (\langle D\rangle^{r-s}\theta_y) * (\langle D\rangle^s f) \in L_p(\mathbb{R}^n)$ since $\langle D\rangle^{r-s}\theta_y \in L_1(\mathbb{R}^n)$. Hence, $f^\sharp(\cdot + iy) \in L_p^\infty(\mathbb{R}^n)$. The last part follows easily from (9.2) and the following lemma, which is a variant of a well-known fact.

Lemma 9.2. *Let $f \in L_1(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} f(x)\,dx = 1$. Set $f_\delta(x) := (1/|\delta|_\star) f(x./\delta)$, where $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n_\star$ and $|\delta|_\star := \prod_{\nu=1}^n |\delta_\nu|$. Then, for every $g \in L_p^s(\mathbb{R}^n)$ $(1 \le p < \infty)$, we have*

$$f_\delta * g \to g \quad in \quad L_p^s(\mathbb{R}^n) \quad as \quad \delta \in \mathbb{R}^n_\star, \delta \to 0\ . \tag{9.3}$$

Proof. First, let $s = 0$. If $g \in \mathscr{S}(\mathbb{R}^n)$, then we have

$$\begin{aligned}(f_\delta * g - g)(x) &= \int_{\mathbb{R}^n} \{f_\delta(y)g(x-y) - f_\delta(y)g(x)\}\,dy \\ &= \int_{\mathbb{R}^n} f(z)\,\{g(x - \delta.{*}z) - g(x)\}\,dz\ ,\end{aligned}$$

and hence, by Lemma 5.4, we have

$$\|f_\delta * g - g\|_{L_p(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} |f(z)|\,\|g(\cdot - \delta.{*}z) - g\|_{L_p(\mathbb{R}^n)}\,dz\ .$$

By continuity, this inequality holds for every $g \in L_p(\mathbb{R}^n)$. Put $G_\delta(z) := \|g(\cdot - \delta.{*}z) - g\|_{L_p(\mathbb{R}^n)}$, then we have $|G_\delta(z)| \le 2\|g\|_{L_p(\mathbb{R}^n)}$. Further, it is well-known that $G_\delta(z) \to 0$ as $\delta \to 0$ for every fixed $z \in \mathbb{R}^n$. Hence, by Lebesgue's dominated convergence theorem, we have $\int_{\mathbb{R}^n} |f(z)|\,G_\delta(z)\,dz \to 0$ as $\delta \to 0$.

For general $s \in \mathbb{R}$, we have only to note that $\langle D\rangle^s(f_\delta * g - g) = f_\delta * (\langle D\rangle^s g) - \langle D\rangle^s g$. □

Next, we prove the case of $M_p^s(\mathbb{R}^n)$. For $y \in \Gamma_\eta$ and for every $r \in \mathbb{R}$, $\langle \xi\rangle^r \widehat{\theta_y}(\xi)$ is bounded on $\mathbb{R}^n$ and hence $f^\sharp(\cdot + iy) \in M_p^\infty(\mathbb{R}^n)$. Note that $\widehat{\theta_y}(\xi)$ is bounded on $\mathbb{R}^n$ uniformly for $y \in \mathbb{R}^n_\star$, since $\sup_{\xi\in\mathbb{R}^n} |\widehat{\theta_y}(\xi)| \le \big(\sup_{\tau\in\mathbb{R}} |\widehat{\theta_0}(\tau)|\big)^n$. Further, $\widehat{\theta_y}(\xi) \to 1$ as $y \to 0$ for every fixed $\xi \in \mathbb{R}^n$. Hence, by Lebesgue's dominated convergence theorem, if $f \in M_p^s(\mathbb{R}^n)$, then we have $\langle\cdot\rangle^s\,\widehat{f}\,\widehat{\theta_y} \to \langle\cdot\rangle^s\,\widehat{f}$ in $L_p(\mathbb{R}^n)$ as $y \to 0$. □

Now, we can use our expansions.

Theorem 9.3. (1) *Let $f \in L_p^s(\mathbb{R}^n)$ $(1 < p < \infty,\ s \in \mathbb{R})$ and set*

$$f_Q := \sum_{j\in\mathbb{Z}} (\mathcal{P}_j^Q f) \in L_p^s(\mathbb{R}^n) \tag{9.4}$$

for $Q \in \mathscr{Q}_\rho$*. Then we have*

$$f(x) = \sum_{Q \in \mathscr{Q}_\rho} \lim_{y \in \Gamma_{\eta(Q)};\, y \to 0} {f_Q}^\sharp(x+iy)\ , \tag{9.5}$$

with convergence in $L^s_p(\mathbb{R}^n)$*. Further,* ${f_Q}^\sharp(z)$ *is expanded as follows, by using the expansion given in Theorem* 4.1*.*

For each fixed $Q \in \mathscr{Q}_\rho$*,* $j \in \mathbb{Z}$*, and* $y \in \mathbb{R}^n$*, the series*

$$\sum_{k \in \mathbb{Z}^n} \langle f, (\psi_Q)_{j,k} \rangle\ {(\psi_Q)_{j,k}}^\sharp(x+iy) \tag{9.6}$$

converges unconditionally in $B_p^{2^jQ}$ *as a function of* x*, and equals* $\left(\mathcal{P}_j^Q f\right)^\sharp(x+iy)$*. For* $y \in \Gamma_{\eta(Q)}$*, the series* $\sum_{j\in\mathbb{Z}} \left(\mathcal{P}_j^Q f\right)^\sharp(x+iy)$ *converges unconditionally in* $L^\infty_p(\mathbb{R}^n)$ *as a function of* x*, and equals* ${f_Q}^\sharp(x+iy)$*.*

(2) *Let* $f \in M^s_p(\mathbb{R}^n)$ $(1 < p < \infty,\ s \in \mathbb{R})$ *and set*

$$f_Q := \sum_{j \in \mathbb{Z}} (\mathcal{P}_j^Q f) \in M^s_p(\mathbb{R}^n) \tag{9.7}$$

for $Q \in \mathscr{Q}_\rho$*. Then we have*

$$f(x) = \sum_{Q \in \mathscr{Q}_\rho} \lim_{y \in \Gamma_{\eta(Q)};\, y \to 0} {f_Q}^\sharp(x+iy)\ , \tag{9.8}$$

with convergence in $M^s_p(\mathbb{R}^n)$*. Further,* ${f_Q}^\sharp(z)$ *is expanded as follows, by using the expansion given in Theorem* 4.1*.*

For each fixed $Q \in \mathscr{Q}_\rho$*,* $j \in \mathbb{Z}$*, and* $y \in \mathbb{R}^n$*, the limit*

$$\lim_{M \to \infty} \sum_{k \in \mathbb{Z}^n; |k_\nu| \le M} \langle f, (\psi_Q)_{j,k} \rangle\ {(\psi_Q)_{j,k}}^\sharp(x+iy) \tag{9.9}$$

exists in $PW_p^{2^jQ}$ *as a function of* x*, and equals* $\left(\mathcal{P}_j^Q f\right)^\sharp(x+iy)$*. For* $y \in \Gamma_{\eta(Q)}$*, the series* $\sum_{j\in\mathbb{Z}} \left(\mathcal{P}_j^Q f\right)^\sharp(x+iy)$ *converges unconditionally in* $M^\infty_p(\mathbb{R}^n)$ *as a function of* x*, and equals* ${f_Q}^\sharp(x+iy)$*.*

Proof. First, we prove (1). The first part of (1) follows easily from Proposition 9.1 (2)(a), since $f_Q \in L^s_p(\mathbb{R}^n)$, $\operatorname{supp} \widehat{f_Q} \subset \overline{\Gamma_{\eta(Q)}}$, and $f = \sum_{Q \in \mathscr{Q}_\rho} f_Q$.

Since 2^jQ is compact, we can take $\phi_y \in \mathscr{S}(\mathbb{R}^n)$ for each $y \in \mathbb{R}^n$ such that $\widehat{\phi_y}(\xi) = e^{-y\cdot\xi}$ in a neighborhood of 2^jQ. Then, ${(\psi_Q)_{j,k}}^\sharp(\cdot + iy) = \phi_y * (\psi_Q)_{j,k}$. Hence, for every finite subset $K \subset \mathbb{Z}^n$, we have

$$\sum_{k \in K} \langle f, (\psi_Q)_{j,k} \rangle \, {(\psi_Q)_{j,k}}^\sharp(\cdot + iy) = \phi_y * \left(\sum_{k \in K} \langle f, (\psi_Q)_{j,k} \rangle \, (\psi_Q)_{j,k} \right) .$$

Since $\phi_y \in L_1(\mathbb{R}^n)$ and since $\sum_{k\in\mathbb{Z}^n} \langle f, (\psi_Q)_{j,k} \rangle \, (\psi_Q)_{j,k}$ converges unconditionally to $\mathcal{P}_j^Q f$ in $B_p^{2^jQ}$ by Theorem 4.1 (1), the series $\sum_{k\in\mathbb{Z}^n} \langle f, (\psi_Q)_{j,k} \rangle \, {(\psi_Q)_{j,k}}^\sharp(\cdot + iy)$ also converges unconditionally to $\phi_y * (\mathcal{P}_j^Q f) = (\mathcal{P}_j^Q f)^\sharp(\cdot + iy)$ in $B_p^{2^jQ}$.

For every finite subset $J \subset \mathbb{Z}$, every $y \in \Gamma_{\eta(Q)}$, and every $r \in \mathbb{R}$, we have

$$\langle D\rangle^r\left(\sum_{j\in J}(\mathcal{P}_j^Q f)^\sharp(\cdot+iy)\right) = \left(\langle D\rangle^{r-s}\theta_y\right) * \left\{\langle D\rangle^s\left(\sum_{j\in J}\mathcal{P}_j^Q f\right)\right\} ,$$

where θ_y is the function introduced in the proof of Proposition 9.1. We also have $\langle D\rangle^{r-s}\theta_y \in \mathscr{S}(\mathbb{R}^n) \subset L_1(\mathbb{R}^n)$ and $\langle D\rangle^s\bigl(\sum_{j\in\mathbb{Z}}\mathcal{P}_j^Q f\bigr)$ converges unconditionally in $L_p(\mathbb{R}^n)$ by Theorem 4.1 (2). Hence, $\sum_{j\in\mathbb{Z}}(\mathcal{P}_j^Q f)^\sharp(\cdot+iy)$ converges unconditionally in $L_p^r(\mathbb{R}^n)$ for every $r \in \mathbb{R}$. Further, we have

$$\sum_{j\in\mathbb{Z}}(\mathcal{P}_j^Q f)^\sharp(\cdot+iy) = \theta_y * \left(\sum_{j\in\mathbb{Z}}\mathcal{P}_j^Q f\right) = f_Q^\sharp(\cdot+iy) .$$

Next, we prove (2). The first part of (2) follows easily from Proposition 9.1 (2)(b), since $f_Q \in M_p^s(\mathbb{R}^n)$, $\operatorname{supp}\widehat{f_Q} \subset \overline{\Gamma_{\eta(Q)}}$, and $f = \sum_{Q\in\mathscr{Q}_\rho} f_Q$.

For every $y \in \mathbb{R}^n$ and $M \in \mathbb{Z}_+$, we have

$$\left(\sum_{k\in\mathbb{Z}^n;|k_\nu|\le M}\langle f,(\psi_Q)_{j,k}\rangle\,(\psi_Q)_{j,k}{}^\sharp(\cdot+iy)\right)^\wedge = e^{-y\cdot\xi}\left(\sum_{k\in\mathbb{Z}^n;|k_\nu|\le M}\langle f,(\psi_Q)_{j,k}\rangle\,(\psi_Q)_{j,k}\right)^\wedge .$$

By Theorem 4.2 (1), we have

$$\left(\sum_{k\in\mathbb{Z}^n;|k_\nu|\le M}\langle f,(\psi_Q)_{j,k}\rangle\,(\psi_Q)_{j,k}\right)^\wedge \to \widehat{\mathcal{P}_j^Q f}$$

in $L_p(\mathbb{R}^n)$ as $M\to\infty$. Since $e^{-y\cdot\xi}$ is bounded on Q, we have

$$\left(\sum_{k\in\mathbb{Z}^n;|k_\nu|\le M}\langle f,(\psi_Q)_{j,k}\rangle\,(\psi_Q)_{j,k}{}^\sharp(\cdot+iy)\right)^\wedge \to e^{-y\cdot\xi}\widehat{\mathcal{P}_j^Q f} = \{(\mathcal{P}_j^Q f)^\sharp(\cdot+iy)\}^\wedge$$

in $L_p(\mathbb{R}^n)$ as $M\to\infty$. Thus, the limit (9.9) exists in $PW_p^{2^jQ}$ and equals $(\mathcal{P}_j^Q f)^\sharp(\cdot+iy)$.

For every finite subset $J \subset \mathbb{Z}$, every $y \in \Gamma_{\eta(Q)}$, and every $r \in \mathbb{R}$, we have

$$\langle\xi\rangle^r\left(\sum_{j\in J}(\mathcal{P}_j^Q f)^\sharp(\cdot+iy)\right)^\wedge(\xi) = (\langle\xi\rangle^{r-s}\widehat{\theta_y}(\xi))\left\{\langle\xi\rangle^s\left(\sum_{j\in J}\mathcal{P}_j^Q f\right)^\wedge(\xi)\right\} .$$

Since $\langle\xi\rangle^{r-s}\widehat{\theta}_y(\xi)$ is bounded as a function of ξ and $\langle\xi\rangle^s\left(\sum_{j\in\mathbb{Z}}\mathcal{P}_j^Q f\right)^\wedge(\xi)$ converges unconditionally in $L_p(\mathbb{R}^n)$ by Theorem 4.2 (2), the series

$$\langle\xi\rangle^r\left(\sum_{j\in\mathbb{Z}}(\mathcal{P}_j^Q f)^\sharp(\cdot+iy)\right)^\wedge(\xi)=\sum_{j\in\mathbb{Z}}\langle\xi\rangle^r\left((\mathcal{P}_j^Q f)^\sharp(\cdot+iy)\right)^\wedge(\xi)$$

converges unconditionally in $L_p(\mathbb{R}^n)$.

Further, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}(\mathcal{P}_j^Q f)^\sharp(\cdot+iy)&=\left\{\left(\sum_{j\in\mathbb{Z}}\mathcal{P}_j^Q f\right)^\wedge(\xi)\,e^{-y\cdot\xi}\right\}^\vee\\&=\left(\widehat{f_Q}(\xi)\,e^{-y\cdot\xi}\right)^\vee=f_Q^\sharp(\cdot+iy)\ .\end{aligned}$$

□

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